The Completion of the Space of Henstock-Bochner Integrable Functions

SERGIO R. CANOY, JR.

In [2], Cao defined the Henstock integral of a Banach-valued function on a compact interval [a,b]. We call such integral the Henstock Bochner integral and denote by H([a,b],X) the space of Henstock-Bochner integrable functions on [a,b] with values in a Banach space X. Also, we denote by E([a,b],X) the space of all X-valued Denjoy integrable functions on [a,b]. It is known that E([a,b],R) = H([a,b],R) where R is the space of real numbers (see ref. [5]). However, Cao in his work showed that for some Banach space X, E([a,b],X) is properly contained in the space H([a,b],X).

The space H([a,b],X) is not complete under the norm given by

$$\|f\|_H = \sup \{\|(HB)\int_a^t f\|;\ a \le t \le b\},$$

(see ref.[5]). Ang, Lee, and Vy in [1] showed that the completion of E([a,b],R) is a subspace of the space of distributions. In this paper, we will show that in general, this result is valid. That is, for any Banach space X, the completion of H([a,b],X) under the norm defined above is a subspace of the space of distributions. In particular, we will show that every Henstock-Bochner integrable function on [a,b] defines a distribution. In order to obtain this result, we need the vector extension of the notion of distributions [8,p 30]. Throughout this paper, X is a real Banach space and O is the zero vector in X.

To proceed, we need the following definitions and results:

Definition 1. A function f : [a,b] —> X is Henstock-Bochner integrable on [a,b] if there is a vector A in X such that for every e > 0 there exists a d(x) > 0 such that for any d-fine division D = {[u,v], x} of [a,b], we have

$$\|(D)\sum f(x(v-u) - A\| < e.$$

SERGIO R. CANOY, JR. is an intructor at the College of Science and Mathematics, MSU-Iligan Institute of Technology. He obtained his Ph.D. in Mathematics at the University of the Philippines, Diliman, Quezon City.

In the above definition, we write

$$(HB)\int_a^b f = A$$

Definition 2. A function f : [a,b] —> X is Denjoy integrable on [a,b] if there exists a function F : [a,b] —> X which is ACG* on [a,b] and such that F′(t) = f(t) almost everywhere in [a,b].

For a more detailed discussion of the above concepts as well as their properties, see refs. [3] and [5].

Definition 3. A function g : [a,b] —> R is said to be of bounded variation on [a,b] if

$$V(g;[a,b]) := \sup\ (D)\Sigma\ |g(v) - g(u)|$$

is finite, where the supremum is over all divisions D = {[u,v]} of [a,b].

Definition 4. Let F : [a,b] —> X and g : [a,b] —> R. We say that F is Henstock-Stieltjes integrable to A (in X) with respect to g on [a,b] if for every e > 0 there exists a d(x) > 0 such that for any d-fine division D = {([u,v], x)} of [a,b], we have

$$\|(D)\Sigma\ (g(v) - g(u))F(x) - A\| < e\ .$$

We remark that if d(x) = n, a constant, for all x in [a,b], then we say that F is Riemann-Stieltjes integrable with respect to g on [a,b]. In any case, we write

$$(HS)\int_a^b Fdg = A\ .$$

The next theorem is known as the Cauchy criterion. The proof is standard (see ref. [3]).

Theorem 5. Let F : [a,b] —> X and g : [a,b] —> R. Then F is Henstock-Stieltjes integrable with respect to g on [a,b] if for every e > 0 there exists a d(x) > 0 such that for any two d-fine divisions D = {[u,v],x} and D′ = {[u′,v′],x′} of [a,b], we have

$$\|(D)\ \Sigma\ (g(v) - g(u))F(x) - (D')\Sigma\ (g(v') - g(u'))F(x')\|_X < e\ .$$

Theorem 6. If F : [a,b] —> X is continuous and g : [a,b] —> R is of bounded variation, then F is Henstock-Stieltjes integrable with respect to g on [a,b].

Proof: Since F is continuous on [a,b], it is uniformly continuous there. That is, given e > 0, there exists n > 0 such that for all t and t′ in [a,b],

$$|t - t'| < n \quad \text{implies} \quad \|F(t) - F(t')\|_X < e.$$

Define d(x) = n/2 for all x in [a,b]. Then for any d-fine divisions D and D′ of [a,b], there exists another d-fine division D″ which is finer than both D and D′. Let [u,v] be a division interval in D. Then

$$u = z_0 < z_1 < \ldots < z_r = v$$

where $[z_{i-1}, z_i]$ is in D″ for i = 1,2,3,...,r. Consider the following difference:

$$\Delta_u^v = [g(v) - g(u)]F(x) - \sum_{k=1}^{r} [g(z_k) - g(z_{k-1})]F(x_k)$$
$$= \sum_{k=1}^{r} [g(z_k) - g(z_{k-1})][F(x) - F(x_k)].$$

Then

$$\|\Delta_u^v\| < e.V(g;[u,v].$$

For D = {[u,v];x}, we write

$$\rho(g,F;D) = (D)\sum (g(v) - g(u))F(x).$$

Therefore,

$$\|\rho(g,F;D) - \rho(g,F;D'')\| < e.V(g;[a,b]).$$

Similarly,

$$\|\rho(g,F;D') - \rho(g,F;D'')\| < e.V(g;[a,b]).$$

Thus,

$$\|\rho(g,F;D) - \rho(g,F;D')\| < 2e.V(g;[a,b]).$$

By the Cauchy criterion, it follows that F is Henstock-Stieltjes integrable with respect to g on [a,b].

The following two results are seen to be similar (but considerably not special cases) to those in the article of Rey and Lee [7]. The first is proved as in the real-valued case (see [5]).

Theorem 7. If f : [a,b] —> X is Henstock-Bochner integrable and g : [a,b] —> R is of bounded variation, then the function g(.)f(.) : [a,b] —> X is Henstock-Bochner integrable on [a,b].

The following result is a quick consequence of the above theorem.

Corollary 9. If f : [a,b] —> X is Henstock-Bochner integrable with primitive F and g : [a,b] —> R is of bounded variation, then

$$(HB)\int_a^b gf = F(b)g(b) - F(a)g(a) - \int_a^b F\,dg .$$

Definition 10. Let D([a,b]) denote the space of all infinitely differentiable real-valued function g : [a,b] —> R with compact support on (a,b). The space of vector distributions, denoted by D′([a,b]), is the space of all continuous linear operators on D([a,b]) taking values in a Banach space X. That is, T is in D′([a,b]) if

$$\langle T, g_n^{(k)}\rangle \longrightarrow \langle T, g^{(k)}\rangle \quad \text{in } X \text{ as } n \longrightarrow \infty$$

whenever $g_n^{(k)} \to g^{(k)}$ uniformly on [a,b] as n goes to infinity. Furthermore, if T is a distribution, its derivative is a distribution DT and

$$\langle DT, g\rangle = -\langle T, g'\rangle \text{ for every } g \in \mathcal{D}([a,b]).$$

Let f : [a,b] —> X be a continuous function. Then it is easy to verify that f defines a distribution. That is, it can be shown that T(f) defined by

$$\langle T(f), g\rangle = \int_a^b gf$$

where the integral is in the Henstock-Bochner sense, is a continuous linear operator on D([a,b]. Here, we also use the fact that a continuous function is Bochner integrable on [a,b] (see ref. [4]). Next, we agree to identify a continuous function with the distribution which it defines. Consequently, we have the following:

Theorem 11. Every continuous function f : [a,b] —> X is differentiable in the distribution sense.

Consider G = { f in D′([a,b]): there exists F in C([a,b],X) with F′ = f in the distribution sense }. Then G is the set of all distributional derivatives of X- valued continuous functions on [a,b]. Since primitives of the element f of G differ only by constant functions, it follows that for every f in G there exists a unique F(f) in C([a,b],X) such that F(f)′ = f in the distribution sense and $F(f)(a) = 0$.

We define the following norm in G:

$$\|f\|_G = \|F(f)\|_\infty = \sup\{\|F(f)(t)\|;\ a < t < b\}.$$

It is not difficult to verify that the above definiton is really a norm in G.

Finally, we have the following result:

Theorem 12. The completion of H([a,b],X) under the norm given earlier is the space G together with the norm defined above. Moreover, the space G is isomorphic to the space C′([a,b],X) with the uniform norm, where C′([a,b],X) = {F in C([a,b],X); F(a) = 0}.

Proof: First, we show that C′([a,b],X) is a closed subspace of C([a,b],X), the space of all X-valued continuous functions on [a,b], under the sup norm. So, let {f(n)} be a sequence of functions in C′([a,b],X) that converge uniformly to f. Then for every e > 0, there exists a natural number N such that

$$\|f(N)(t) - f(t)\|_X < e$$

for all t in [a,b]. It follows that the norm of f(a) in X is less than e for every $e \geq 0$. Hence, f(a) = 0 and we have the desired result. Thus C′([a,b],X) under the sup or uniform norm is a Banach space.

Now, define the following mapping:

$$T : (G, \|\cdot\|_G) \longrightarrow (C'([a,b],X), \|\cdot\|_\infty) \quad \text{by}$$

$$T(f) = F(f).$$

Then T is linear and injective. Also, if F is in C′([a,b],X), then F′ = f (in the distribution sense) is in G and T(f) = F. Hence T is surjective. Moreover, T is norm preserving since

$$\|f\|_G = \|F(f)\|_\infty = \|T(f)\|_\infty$$

Therefore, T is an isomorphism. It follows that G under the defined norm is a Banach space and T is a Banach isomorphism.

Next, let f: [a,b] —> X be a Henstock-Bochner integrable function on [a,b] with primitive F given by

$$F(t) = (HB)\int_a^t f$$

Clearly, F(a) = 0. Also, F is continuous (see [2]). Define

$$<f,g> = (HB)\int_a^b gf \quad \text{for } g \in \mathcal{D}([a,b]).$$

Note that the existence of the above definition follows from Theorem 7. Now, in view of Corollary 8,

$$<f,g> = F(b)g(b) - \int_a^b F dg = - <F,g'>, \; g \in \mathcal{D}([a,b]).$$

Thus,

$$\|<f,g>\| \le \|F\|_\infty |g|_{C^1([a,b],R)}.$$

It follows that f is a distribution and F is a distributional primitive of f. Furthermore, we see that f is in G, F(f) = F and

$$\|f\|_H = \|F\|_\infty = \|F(f)\|_\infty = \|f\|_G.$$

This means that

$$(H([a,b],X), \|\cdot\|_H) \subset (G. \|\cdot\|_G).$$

Next, we will show that E([a,b],X) is dense in G. Let f be an element of G. Since F(f) is continuous, there exists a sequence {F(n)} of piecewise linear functions (hence ACG*) such that F(n) ---> F(f) uniformly on [a,b]. Without loss of generality, we may assume further that F(n)(a)=0 for all n. Then f(n)=F(n)' is in E([a,b],X), F(n)=F(f(n)) for all n and

$$\|f(n) - f\|_G = \|F(n) - F(f)\|_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty ,$$

i.e., f(n) ---> f in G as n goes to infinity. This shows that E([a,b],X) is dense in G. Accordingly, H([a,b],X) is also dense in G.

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