A Short Note on the SL-Integral

SERGIO R. CANOY, JR.

In [2], Lee introduced the strong Lusin condition and in [4], both Vyborny and Lee defined and studied the SL-integral for real-valued functions on the compact interval [a,b].

Rey, in his work [3], extended the above concepts to the vector-valued case. He was able to show that some of the results in the real-valued case hold naturally in the vector-valued case. However, he never attempted to give the vector version of the following result found in [4]:

(*) If f:[a,b] —> R is Lebesgue integrable on [a,b], then f is SL-integrable there and

$$(SL)\int_a^b f = (L)\int_a^b f$$

The natural vector extension of the Lebesgue integral is the Bochner integral. Hence, in this very short note, we will show that if f : [a b] —> X, where X is a Banach space, is Bochner integrable on [a,b], then it is SL-integrable there.

To proceed we need the following definitions:

Definition 1. A function F : [a,b] —> X is said to satisfy the Strong Lusin condition on [a,b] if for every set E of measure zero and every e > 0, there exists a function d(x) > 0 on E such that for any d-fine partial division D = {[u,v],x} of [a,b] with x in E, we have

$$(D)\Sigma\|F(v) - F(u)\|_X < e.$$

In this case, we call F an SL-function.

Definition 2. A function f : [a,b] —> X is said to be SL-integrable on [a,b] if there exists an SL-function F from [a,b] into X and for every e > 0 there exists a nonnegative function (or a gauge) d on [a,b] such that for every d-fine partial division D = {[u,v];x} of [a,b], we have

$$\|(D)\Sigma\{f(x)(v-u) - F(v) + F(u)\}\|_X < e.$$

SERGIO R. CANOY, JR. is an intructor at the College of Science and Mathematics, MSU-Iligan Institute of Technology. He obtained his Ph.D. in Mathematics at the University of the Philippines, Diliman, Quezon City.

In this case, we write

$$(SL)\int_a^b f := F(b) - F(a).$$

For more details about the SL-integral, see [3] and [4].

Definition 3. A function $F : [a,b] \longrightarrow X$ is said to be differentiable at t' in (a,b) if

$$\lim_{t \to t'} \frac{F(t) - F(t')}{t - t'}$$

exists in X. We denote this limit, if it exists, by $F'(t')$. If F is differentiable at all t on (a,b), then F is said to be differentiable on (a,b).

Definition 4. A measurable function $f : [a,b] \longrightarrow X$ is said to be Bochner-integrable on [a,b] if there exists an absolutely continuous function $F : [a\ b] \longrightarrow X$ such that F is differentiable a.e. on [a b] and $F'(t) = f(t)$ a.e. on [a b].

We now state and prove the vector version of (*) given above.

Theorem 4. If $f : [a,b] \longrightarrow X$ is Bochner integrable on [a,b], then f is SL-integrable there.

Proof: Let F be the Bochner primitive of f. Then F is absolutely continuous and $F'(t) = f(t)$ a.e. on [a,b]. Let E be a subset of [a,b] with measure zero and let $e > 0$. Since F is absolutely continuous, there exists an $n > 0$ such that for every sequence of non-overlapping intervals $\{[a_i, b_i]\}$ of [a,b],

$$\Sigma_i (b_i - a_i) < n \quad \text{implies} \quad \Sigma_i \|F(b_i) - F(a_i)\|_X < e.$$

Also, since E is of measure zero, then there exists a sequence of open intervals such that E is contained in the union of these intervals and the sum of the lengths of the intervals is less than n. Define $d(x) > 0$ so that $(x-d(x), x+d(x))$ is contained in the union of the intervals if x is in E and arbitrarily if otherwise. Therefore if $D = \{[u,v]; x\}$ is any d-fine partial division of [a,b] with x in E, then

$$(D)\Sigma \|F(v) - F(u)\|_X < e.$$

Therefore F is an SL-function.

Next, let E′ consist of all t in [a,b] such that F′(t) does not exist, or, if it does, F′(t) is not f(t). Then by definition, E′ is of measure zero. Since F is an SL-function, given e > 0 there exists a d(x) > 0 defined on E′ such that for any d-fine partial division D′ = {[u,v]; x] of [a,b] with x in E′, we have

$$(D')\Sigma\|F(v) - F(u)\|_X < e.$$

Now, for each x in [a,b]\ E′, there exists d′(x) > 0 such that if x is in [u,v] and [u,v] is contained in (x-d′(x),x+d′(x)), we have

$$\|f(x)(v-u) - F(v) + F(u)\|_X \leq e(v-u).$$

Define a function g : [a,b] —> X as follows:

g(t) = f(t) if t is in E′ and g(t) = 0 if otherwise.

Then g = 0 a.e. on [a,b]. Thus g is Henstock-Bochner integrable to the zero vector on [a,b] (see [1]). Hence for the given e > 0, there exists d″(x) > 0 such that for any d″-fine partial division D″= {[u,v];x} of [a,b], we have

$$\|(D'')\Sigma\, g(x)(v-u)\|_X < e.$$

Define d‴(x) > 0 on [a,b] as follows:

d‴(x) = min {d(x),d″(x)}, if x is in E′ and
d‴(x) = d′(x) , if x is not in E′.

Therefore if D = {[u,v];x} is a d‴-fine partial division of [a,b], then

$$\|(D)\Sigma\{ f(x)(v-u) - F(v) + F(u) \}\|_X$$

$$\leq \|\sum_{x\notin E'} \{ f(x)(v-u) - F(v) + F(u) \}\|_X$$

$$+ \|\sum_{x\in E'} f(x)(v-u)\|_X + \sum_{x\in E'} \|F(v) - F(u)\|_X$$

$$< e(b-a) + e + e = (b-a+2)e$$

Therefore f is SL-integrable on [a,b].

References

Cao, S.S., The Henstock Integral for Banach-Valued Functions, *SEA Bull. Math.* 16 No.1 (1992) 35-40.

Lee, P.Y., On ACG Functions, Real Analysis Exchange 15 (1989- 1990), 754-759.

Rey, R.M., The Henstock-Bochner Integral and Its Extensions, Ph.D. dissertation, UP-Diliman, 1992.

Vyborny, R and Lee, P.Y., Kurzweil-Henstock Integration and the Strong Lusin Condition, Research Report, No.458 Lee Chian Centre for Math'l Research, NUS , 1991.