On ACG* and the Strong Lusin Condition

SERGIO R. CANOY, JR.

Lee introduced two important concepts in the Theory of Henstock integration namely : the ACG* and the the Strong Lusin (SL) condition (see [3] and [5]). The former is used to define the Denjoy integral and in stating the controlled-convergence theorem while the latter is used to define the SL integral (see [2] and [4]).

In [3], Lee gave an alternative definition of the Denjoy integral by means of the SL condition. Although the SL-condition simplifies slightly the definition of the Denjoy integral, it is still necessary to adopt the definition of ACG* in stating some important results like the controlled-convergence theorem.

In this note, we will state and prove some results concerning the ACG* and the SL-condition for Banach-valued functions. Throughout this paper, X will be used to denote a real Banach space.

Let us recall some important definitions:

Definition 1. A function F : [a b] —> X is said to be differentiable at x′ in (a b) if

$$\lim_{x \to x'} \frac{F(x) - (x')}{x - x'}$$

exists in X. We denote this limit, if it exists, by F′(x′). If F is differentiable at all x in (a,b), then F is said to be differentiable on (a,b).

Definition 2. A function F : [a b] —> X is said to satisfy the Strong Lusin (SL) condition if for every subset E of [a,b] of measure zero and for every e > 0, there exists a d(x) > 0 on E such that for any d-fine partial division D = {[u v]; x} with x in E, we have

$$(D)\Sigma\|F(u,v)\|_X < e$$

where F(u,v) = F(v) - F(u). Recall that a division D = {[u,v]; x} of [a,b] consisting of interval-point pairs is a d-fine division of [a,b] if x is in [u,v] and [u,v] is contained in (x-d(x),x+d(x)).

Definition 3. Let E be a subset of [a,b]. A function F from [a,b] into X is said to be AC*(E) if for every e > 0, there exists an n > 0 such that for any partial division D = {[u,v]} of [a,b] with u or v in E ,

---

SERGIO R. CANOY, JR. is an intructor at the College of Science and Mathematics, MSU-Iligan Institute of Technology. He obtained his Ph.D. in Mathematics at the University of the Philippines, Diliman, Quezon City.

$$(D)\Sigma|v-u| < n \text{ implies } (D)\Sigma\|F(v)-F(u)\|_X < e.$$

Definition 4. A function F : [a,b] —> X is said to be ACG* on [a,b] if [a,b] is the union of sets Y(i), i = 1, 2 ... such that F is AC*(Y(i)) for each i.

Definition 5. A function f : [a,b] —> X is HL-integrable on [a,b] if there exists an additive function F : [a,b] —> X, called the primitive of f, satisfying the following property : For every e > 0, there exists a d(x) > 0 such that for any d-fine division D = {[u,v]; x} of [a,b], we have

$$(D)\Sigma\|f(x)(v-u) - F(v) + F(u)\|_X < e.$$

Definition 5 is due to Cao [1]. We recall that in the real-valued case, the Henstock integral and the HL-integral are equivalent. However, this is not true for the Banach-valued case. For our purpose here, we need to state and prove the following first important result:

Lemma 6. Let f : [a b] —> X be an HL-integrable function on [a,b] with primitive F : [a b] —> X. Then F is ACG* on [a,b].

Proof: Let e > 0. Since f is HL-integrable on [a,b], there exists a d(x) > 0 on [a,b] such that for any d-fine division D = {[u,v]; x} of [a,b], we have

$$(D)\Sigma\|f(x)(v-u) - F(v) + F(u)\|_X < e.$$

Let E(i) consist of all t in [a,b] such that that the norm of f(t) on X is less than or equal to i and d(t) > 1/i. Clearly, [a,b] is the union of the sets E(i), i = 1,2,...

It remains to show that F is AC*(E(i)) for each i. Define n = min { e, 1/i }. Let D′= {[u,v]} be any partial division of [a,b] with u or v in E such that the sum of the lengths of the intervals in D′ is less than n. For each [u,v], choose x in [u,v] with x in E(i) (for example, x = u if u is in E(i) or x = v if v is in E(i)). Thus D′= {[u,v]; x} is a d-fine partial division of [a,b]. Therefore

$$\begin{aligned}
&(D')\Sigma\|F(v)-F(u)\|_X \\
&\le (D')\Sigma\|F(v) - F(u) - f(x)(v-u)\|_X \\
&\quad + (D')\Sigma\|f(x)(v-u)\|_X \\
&< e + i\,(D')\Sigma|v-u| \\
&< (1 + i)e.
\end{aligned}$$

This shows that F is AC*(E(i)) for each i. Therefore F is ACG* on [a,b].

We now state and prove our first main result.

Theorem 7. If F : [a,b] —> X is differentiable almost everywhere on [a,b] and satisfies the Strong Lusin condition then F is ACG* on [a,b].

Proof: Let E = { t in [a,b]: F′(t) does not exist}. Then E is of measure zero.

Next, we define a function f : [a,b] —> X as follows:

$$f(t) = F'(t), \text{ if } t \text{ is in } [a,b] \setminus E \text{ and}$$
$$f(t) = 0 \quad , \text{ if } t \text{ is in } E.$$

We claim that f is HL-integrable on [a,b] with primitive F. So, let e > 0. By definition of differentiability, there exists a d′(x) > 0 such that whenever x is in [u,v] and [u,v] is contained in (x-d′(x),x+d′(x)) and x is in [a,b]\ E, we have

$$\|f(x)(v-u) - F(v) + F(u)\|_X < e(v-u).$$

Also, since E has measure zero and F satisfies the SL-condition, there exists a d″(x) > 0 on E such that for any d″-fine partial division D″= {[u,v]; x} with x in E, we have

$$(D'')\Sigma\|F(v) - F(u)\|_X < e.$$

Define d(x) > 0 on [a,b] as follows:

$$d(x) = d'(x), \text{ if } x \text{ is in } [a,b]\setminus E$$
$$d(x) = d''(x), \text{ if } x \text{ is in } E.$$

Therefore, if D = {[u,v]; x} is a d-fine division of [a,b],then

$$(D)\Sigma\|f(x)(v-u) - F(v) + F(u)\|_X$$
$$\leq \sum_{x\notin E}\|f(x)(v-u) - F(v) + F(u)\|_X$$
$$+ \sum_{x\in E}\|F(v) - F(u)\|_X < e(b-a) + e.$$

Therefore, f is HL-integrable on [a,b] with primitive F. In view of Lemma 6, F is ACG* on [a,b].

In what follows, we give our last main result:

Theorem 8. If F : [a b] —> X is ACG* on [a,b], then F satisfies the

Strong Lusin condition.

Proof: Let E be a subset of [a,b] with measure 0 and let $e > 0$. Since F is ACG* on [a,b], it follows that [a,b] is the union of sets E(i) such that F is AC*(E(i)) for each i = 1, 2,... Let Y(1) = E(1) and Y(i) = E(i)\(E(1)UE(2)U...UE(i-1)) for i = 2, 3... Let S(i) be the intersection of the sets E and Y(i) for each i = 1,2,... Then E is the union of the sets S(i) and S(i) has measure 0 for each i. Also, the sets S(i) are pairwise disjoint. Since F is also AC*(S(i)), there exists $n(i) > 0$ such that for any partial division D = {[u,v]} of [a,b] with u or v in S(i),

$$(D)\Sigma|v-u| < n(i) \text{ implies } (D)\Sigma\|F(v) - F(u)\|_X < e2^{-i}.$$

Furthermore, since S(i) has measure 0, there exists an open set G(i) which is the union of countable number of open intervals such that S(i) is contained in G(i) and the measure of G(i) is less than n(i). Define $d(x) > 0$ so that (x-d(x),x+d(x)) is contained in G(i) when x is in S(i) and arbitrarily otherwise. Therefore, if D = {[u,v]; x} is any d-fine partial division of [a,b] with x in E, then

$$(D)\Sigma\|F(v) - F(u)\|_X = \sum_{x\in S(i)} \|F(v) - F(u)\|_X$$
$$< \sum_i e2^{-i} = e.$$

Therefore, F satisfies the Strong Lusin condition.

The following result follows directly from Theorems 7 and 8:

Theorem 9. Let F : [a b] —> X be a function which is differentiable almost everywhere on [a,b]. Then F is ACG* on [a,b] if and only if it satisfies the Strong Lusin condition.

References

Cao, S.S., The Henstock Integral for Banach-valued Functions *SEA Bull. Math.* 16 No.1(1992) 35-40.

Lee, P.Y., Lanzhou Lectures in Henstock Integration, *World Scientific*, 1989.

Lee, P.Y., On ACG* Functions, Real Analysis Exchange 15 (1989-90) 754-759.

Rey, R.M., The Henstock-Bochner Integral and It's Extensions, Ph.D. Dissertation UP-Diliman, 1992.

Vyborny, Ř. and Lee, P.Y., Kurzweil-Henstock Integration and the Strong Lusin Condition, R.R. No. 458, Lee Kong Centre for Math'l. Research, NUS, 1991.