SENSITIVITY ANALYSIS ON THE OBJECTIVE FUNCTION COEFFICIENTS OF LINEAR PROGRAMMING PROBLEMS

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Introduction

For the general linear programming problem

$$\begin{aligned} \text{Maximize } CX &= z \\ \text{such that } AX &= b \ (*) \\ X &\geq O \end{aligned}$$

C is the objective function coefficient matrix, X is the column matrix corresponding to the basic and non-basic variables, A is the coefficient matrix of X and b is the right-hand side constants. $Z = CX$ is called the objective function while $AX = b$, $x \geq O$ are the constraints of the linear model.

The simplex method of Dantzig [1] provides an effective way to solve for the values of X which maximizes the objective function in (*) and at the same time satisfying the given constraints. A detailed discussion of the steps in the simplex method is made to facilitate the reader's understanding of this article. We shall call the Cj, A_{ij}, b_i as the parameters. It is assumed in Linear Programming that these parameters should be constants and accurately known in advance. However, this sounds unrealistic since in most cases this assumption is never fully satisfied. These parameter values usually vary frequently, significantly and independently, as in the feed-mix (blending) problem.

Considering price fluctuations in our unstable market today and assuming that the constraints are fixed, chances are that the current optimal solution to a given feed-mix problem would be affected. Given this situation, a decision-maker would like to determine the range within which the parameter of the problem can be altered without affecting the optimal solution. This can only be done through a method known as Sensitivity Analysis or Parametric Programming. Hence, the objective of this paper is to demonstrate the procedures of Sensitivity Analysis on the Coefficients of the objective function of (*) by the primal-dual approach.

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The Simplex Method

The three basic steps of the simplex method are the following:

Initialization step	:	Identify an initial basic feasible solution.
Iterative step	:	Move to a better adjacent basic feasible solution.
Stopping rule	:	Stop when no adjacent basic feasible solution is better.

Let us consider one example of linear programming problem and apply the three steps above to obtain the optimal solution.

$$\text{Maz } Z = 3X_1 + 5x_2$$

Subject to:

$$X_1 \leq 4$$

$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

$$x_1, x_2 \geq 0$$

Initialization step: Introduce slack variables (S_1, S_2, S_3), then select the original variables (X_1, X_2) to be the initial non-basic variables, set equal to zero, and the slack variables to be the initial basic variables.

When solving a problem by hand, it is more convenient to use the tabular form. Instead of writing down each set of equation in detail, simply use a simplex tableau, recording only the essential information, namely: (1) the coefficients of the variables, (2) the constants on the right-hand side of the operations, and (3)the basic variable appearing in each equation. The initial simplex tableau of the example is shown in Table I.

Table I. Initial Simplex Tableau

Basic variable	Z	Coefficient of					Constant Column
		X_1	X_2	S_1	S_2	S_3	
S_1	0	1	0	1	0	0	4
S_2	0	0	2	0	1	0	12
S_3	0	3	2	0	0	1	18
$(C_j - Z_j)$ Max	−1	3	5	0	0	0	0

Table I shows that the initial basic feasible solution is (0, 0, 4, 12, 18). Go next to the stopping rule to determine if this solution is optimal.

Stopping rule: The current basic feasible solution is optimal if and only if every coefficient in $(C_j - Z_j)$ row is negative or zero ($\leqslant 0$). If it is, stop: otherwise go to the iterative step to obtain the next feasible solution– which involves changing one nonbasic variable to a basic variable and vice-versa, and then compute for the new solution.

The example has two positive coefficients in the $C_j - Z_j$ row, 3 for x_1 and 5 for x_2, so go to the iterative step.

Iterative step:

Part I. Determine the entering basic variable by selecting the non-basic variable with the largest positive coefficient in the $C_j - Z_j$ row. (This is the non-basic variable that would increase Z at the fastest rate by being increased from zero). Put a box around the column above this coefficient, and call this the pivot column.

In the example, the largest positive coefficient is 5, so x_2 becomes a basic variable.

Part II. Determine the leaving basic variable by:

(a) picking out each coefficient in the boxed column that is strictly positive (>0);

(b) dividing each of these coefficients into the constants on the right side for the same row;

(c) identifying the equations that have the smallest of these ratios; and

(d) selecting the basic variable for this equation. (This is the basic variable that reaches zero first as the entering basic variable is increased). Put a box around this equation's row in the tableau and call the boxed row the pivot row. Call the number in the intersection of the two boxes the pivot number.

The result of Parts I and II for the example (before placing a box on the row) is shown in Table II. Thus, the leaving basic variable is S2.

Part III. Determine the new basic feasible solution by constructing a new simplex tableau below the current one. The leaving basic variable in the first column is replaced by the entering basic variable. The coefficient of the new basic variable should be changed to +1 by dividing the entire pivot row (i.e. every number in that row including the right side) by the pivot number, so that:

$$\text{New pivot row} = \frac{\text{old pivot row}}{\text{pivot number}}$$

To eliminate the new basic variable from the other equations, every row except the pivot row is changed for the new tableau by using the following formula:

New row = old row − (pivot cal. coefficient) x new pivot row, where "pivot column coefficient" is the number in this row that is in the pivot column.

Table II Calculation to show the first leaving variable.
Iteration 0

Basic Variable	Z	X_1	X_2	S_1	S_2	S_3	Constant Column
			Coefficient of				
S_1	0	1	0	1	0	0	4
S_2	0	0	2	0	1	0	12− $\frac{12}{2}$ = 6
S_3	0	3	2	0	0	1	18 18/2 = 9
$(C_j - Z_j)$ Max	−1	3	5	0	0	0	0

To illustrate, the new rows for the example are obtained below:

Row 1 is unchanged because its pivot column coefficient is 0.

Row 2 New Row 2 = $\dfrac{\text{old Row 2}}{\text{pivot number}}$

= 0 0 1 0 1/2 0 6

Row 3 [0 3 2 0 0 1 18]

−2 [0 0 1 0 1/2 0 6]

New R_3 = 0 3 0 0 −1 1 6

$(C_j - Z_j)$ row [−1 3 5 0 0 0 0]

−5 [0 0 1 0 1/2 0 6]

New $(C_j - Z_j)$row −1 +3 0 0 −5/2 0 −30

This yields the new tableau shown in Table III for Iteration 1.

Table III. Iteration 1

Basic Variable	Z	X_1	X_2	Coefficient of S_1	S_2	S_3	
S_1	0	1	0	1	0	0	4
X_1	0	0	1	0	1/2	0	6
S_3	0	3	0	0	–1	1	6
$(C_j - Z_j)$ Max	–1	3	0	0	–5/2	0	–30

Table III shows that the new basic feasible solution is (0, 6, 4, 0, 6), with Z = 30.

This completes the iterative step. Next, turn to the stopping rule to check whether the new solution is optimal. Since the $C_j - Z_j$ row still has a +3 coefficient, the stopping rule indicates that the solution is not optimal.

Repeating the iterative steps, X_1 enters the basis while S_3 leaves $(6/3 \leq \frac{4}{1})$ the basis. We then apply the pivot rule to obtain Table IV.

Table IV. Iteration II

Basic Variable	Z	X_1	Coefficients of X_2	S_1	S_2	S_3	Constant column
S_1	0	0	0	1	1/3	–1/3	2
XX_2	0	0	1	0	1/2	0	6
X_1	0	1	0	0	– 1/3	1/3	2
$(C_j - Z_j)$ Max	–1	0	0	0	–3/2	–1	–36

Since the $C_j - Z_j \leq 0$, Table IV is an optimal simplex tableau the optimal solution is (2, 6, 2, 0, 0) with Z = 36.

Primal-Dual of Linear Programming

Given a maximization linear programming problem, there corresponds a minimization problem which has the same solution as the original one. The first problem is called the primal while the latter is its dual. Similarly, if the primal is a minimization problem, then the dual should be a maximization one.

It is advantageous to know the primal-dual relationship since in some cases it is more convenient to obtain the solution of the primal through the dual if the number of constraint equations of the dual is less is than that of the primal. This can be attributed to the fact that the number of iterations (simplex tableaus) required to obtain an optimal solution depends on the number of equations and not on the number of variables.

The primal-dual relationship can best be illustrated by means of a simple example on blending problem. This will facilitate the presentation of principles and computation procedures in a manner which will hasten the reader's understanding.

An Illustrative Example

Consider a chemical company which produces two liquid fertilizers – Alfa phosphate and Beta phosphate (X_1 and X_2). These two fertilizers are produced from a blend of three kinds of crude oil: Light Arabian, Basra, and Nigerian. The production requirements of the two fertilizers and the maximum available quantity of each crude oil are shown in Table 1. Regarding the selling prices, the company has set them so as to realize contribution margins of 30¢ and 20¢ per gallon of Alfa phosphate and Beta phosphate, respectively. The problem which this company faces is the determination of the optimum fertilizer mix that maximizes the total daily contribution margin. [4]

Table 5. Production Requirements for Alfa Phosphate and Beta Phosphate Fertilizers

Crude Oil	Production Requirements per Gallon of Alfa Phosphate (X_1) from Crude Oil	Production Requirements per Gallon of Alfa Phosphate (X_2) from Crude Oil	Maximum Available Quantity in Gallons per Day
Light Arabian	1	3	1,200
Basra	3	4	3,000
Nigerian	8	4	4,000

Mathematically, we can formulate the above problem as follows:

$$\text{Maximize } p = 30x_1 + 20x_2$$

$$\begin{aligned} \text{s.t.} \quad x_1 + 3x_2 &\leqslant 1{,}200 \\ 3x_1 + 4x_2 &\leqslant 3{,}000 \\ 8x_1 + 4x_2 &\leqslant 4{,}000 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$

A. Let S_1, S_2, S_3 be the slack variables (non-basic), the matrices C =
C = [30, 20, 0, 0, 0];

$$X = \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} ; \qquad b = \begin{bmatrix} 1{,}200 \\ 3{,}000 \\ 4{,}000 \end{bmatrix} ;$$

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 \\ 8 & 4 & 0 & 0 & 1 \end{bmatrix}$$

B. The dual of this maximization problem is
minimize $C = 1200y_1 + 3000y_2 + 4000y_3$

$$\begin{aligned} \text{s.t.} \quad y_1 + 3y_2 + 8y_3 &\geqslant 30 \\ 3y_1 + 4y_2 + 4y_3 &\geqslant 20 \\ y_1, y2, y_3 &\geqslant 0 \end{aligned}$$

The following are the steps in changing a primal problem to a dual:

(1) Transpose the columns in the primal coefficient matrix to rows in the dual coefficient matrix.
(2) Replace the variable x_j in the primal by Y_1 in the dual.
(3) The coefficients of the objective function of the primal should become the constants of the dual equations.
(4) The constants of the primal equation should become the objective function coefficients of the dual.
(5) The direction of the inequalities of the dual should be opposite to that of the primal.

C. To solve the primal problem above by the simplex method, we add the 3 slack variables S_1, S_2, S_3, and the problem comes:

Maximize: $P = 30x_1 + 20x_2 + Os_1 + Os_2 + OS_3$

s.t.
$$x_1 + 3x_2 + S_1 + OS_2 + OS_3 = 1200$$
$$3x_1 + 4x_2 + Os_1 + S_2 + OS_3 = 3000$$
$$8x_1 + 4x_2 + OS_1 + OS_2 + S_3 = 4000$$

$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

The optimal tableau for the primal problem is shown in Table II.

Table VI

Basis	p	x_1	x_2	S_1	S_2	S_3	(b) Constant Column
x_2	0	0	1	2/5	0	–1/20	280
S_2	0	0	0	–1	1	–1/4	800
x_1	0	1	0	–1/5	0	3/20	360
$(C_j - Z_j)$	–1	0	0	– 2	0	– 7/2	–16400

From Table VI, the primal optimal solutions are

$x_2 = 280$
$S2 = 800$ $S_1 = 0$
$x_1 = 360$ $S_3 = 0$

Maximum profit: 16,400 (cents)

From Table VI, we also pick up the optimal solutions for the dual problem by scanning the values in the $C_j - Z_j$ row as follows:

$Y_1 = 2$

$Y_2 = 0$

$Y_3 = 7/2$

D. Now, we can check our results in C by solving the dual problem. Let S_1 and S_2 be the surplus variables and A_1 and A_2 be the artificial variables. So the dual problem is:

$$\text{Minimize } C = 1{,}200y_1 + 3{,}000y_2 + 4{,}000y_3 - 0S_1 - 0S_2 + MA_1 + MA_2$$

$$\text{s.t. } y_1 + 3y_2 + 8y_3 - S_1 + A_1 = 30$$

$$3y_1 + 4y_2 + 4y_3 - S_2 + A_2 = 20$$

$$y_1, y_2, S_1, S_2, A_1, A_2 \geq 0$$

Using the big-M method in [6], the optimal tableau for the dual problem is shown below in Table 3.

Table VII

Basis	C	y_1	y_2	y_3	S_1	S_2	A_1	A_2	(b) Constant col.
y_3	0	0	1/4	1	−3/20	1/20	3/20	−1/20	7/2
y_1	0	1	1	0	1/5	−2/5	−1/5	2/5	2
$(C_j - Z_j)0$ Min	1	0	800	0	360	280	M-360	M-280	16400

From Table VII above, the optimal dual values are:

$y_1 = 2$

$y_2 = 0$ Min. C = 16,400 (cents)

$y_3 = 7/2$

Scanning through the $C_j - Z_j$ row, the primal solutions

are: $x_1 = 360$

$x_2 = 280$ $S_2 = 800$

It is very apparent that the solutions obtained in D are consistent with those in C.

E. Sensitivity Analysis on the coefficient of the objective Functions of the Primal.

First, it is necessary to determine whether the coefficient under consideration is for a basic or a non-basic variable. A basic variable is one which appears in the Basis columns, otherwise, it is non-basic.

For our primal problem P_1 (30) is a coefficient of a basic variable x_1. We are interested in the query of how much the cost of x_1 can increase (decrease) without affecting the current variables in the basis. Since the primal is a maximization one, the variables in the basis will not be altered so long as the $c_j - z_j \leq 0$.

Suppose we let DP_1 represent the change on P_1 of the variable x_1, the new P_1 will be $P_1^* = P_1 + DP_1$ ($P_1^* = 30 + D$). TableIV shows the final simplex tableau after changing the P_1 of x_1 by D.

Table VIII

Basis	P	x_1	2_x	S_1	S_2	S_3	Constant (b)
x_2	0	0	1	2/5	0	−1/20	280
S_2	0	0	0	−1	1	−1/4	800
x_1	0	1	0	−1/5	0	3/20	360
$(c_j - Z_j)$ Min	1	0	0	−2+D/5	0	$-7/2 - {}^{30}/20$	−16400

The solutions of Table VIII will only be optimal if all the $C_j - Z_j \leq 0$. Thus, D should satisfy the following inequalities:

$$-2 + (D/5) \leq 0 \quad \text{and} \quad -7/2 \quad -(3D/20) \leq 0$$

Solving the above inequalities algebraically, we find the limits of D that satisfy them as: $\frac{-70}{3} \leq D \leq 10$. This result can be interpreted as: the coefficient of the profit of x_1 can lincrease by as much as 10 and decrease by 23 (1/3) and yet the optimal solution is not changed in the basis.

F. If we work on the optimal tableau of the dual, we will find out that the result is identical. To summarize, one can determine the range of optimality of any linear programming parameter from the primal or the dual tableau by the following procedures:

Primal Tableau Range of Optimality of		Dual Tableau Range of Optimality of
b1	corresponds to	c_j
c_j	corresponds to	b_i

REMARKS

This paper has so far demonstrated the primal-dual relationship and its role to sensitivity analysis. The discussion was centered on the perturbation of the c_j parameter which is the coefficient of the objective function.

The method and procedure discussed above has its limitations. First, if changes on the cost coefficients will not fall on the sensitivity analysis ranges, a decision-maker has to do computer reruns especially for large programs to be able to obtain a new set of optimal solutions. This sounds quite costly. Secondly, in conducting sensitivity analyses on the coefficients of the objective function, we have to hold other parameters constant.

The recent publication of Prof. Ralph E. Steuer entitled "Algorithms of Linear Programming Problems with Interval Objective Function Coefficients," offers a remedy to the limitations stated above. In this article, he presented three algorithms which solve (*) where c-e [α/N]. These algoriths output all extreme points and unbounded edge directions that are "multiparametrically optimal" with respect to the ranges placed on the objective function coefficients. If the interval [α/N] is the sensitivity analysis range that preserves the optimality of the generated solution, these algorithms would provide the decision-maker with the optimal solution desired with due considerations to variations on the parameter C_j. However, the presentation of these algorithms is not within the scope of this article.

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