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Equi-integrability in the Harnack Extension

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Abstract: In this note, we provide an alternative proof of the Harnack Extension by means of equi-integrability.

Keywords/Phrases: Henstock integral, equi-integrable, uniformly gauge Cauchy, Harnack Extension

1 Introduction

In the classical theory of Denjoy-Perron integral, it contains basically one proof, the proof by category argument. See for example, Lee [3, p.47] and Saks [7, p.253]. In fact, when Denjoy first defined his integral, which is known to be equivalent to the Henstock integral, he used transfinite induction to extend the Lebesgue integral by means of Cauchy and Harnack extensions [7]. In [5], Lee also extends the Harnack extension for the Henstock integral on $\mathbb{R}$ to $\mathbb{R}^n$ so that it is real-line independent.

Using the concept of equi-integrability, we provide an alternative means to prove the Harnack Extension for the Henstock integral on the real line.

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2 Preliminary Results

Definition 2.1 A function $f : [a,b] \to \mathbb{R}$ is said to be *Henstock integrable to a real number* A *on* $[a,b]$ if for any $\epsilon > 0$, there exists a function $\delta : [a,b] \to \mathbb{R}^+$ such that for any Henstock δ-fine division $D = \{([u,v],\xi)\}$ of $[a,b]$, we have

$$\left|(D)\sum f(\xi)(v-u) - A\right| < \epsilon.$$

If $f : [a,b] \to \mathbb{R}$ is Henstock integrable to A on $[a,b]$, then we write

$$A = (\mathcal{H})\int_a^b f.$$

By a Henstock δ-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$ we mean that $\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$, for all $([u,v];\xi) \in D$.

Lemma 2.2 (Henstock's Lemma) *If* $f : [a,b] \to \mathbb{R}$ *is Henstock integrable on* $[a,b]$ *with primitive* F*, then for each* $\epsilon > 0$ *there exists* $\delta : [a,b] \to \mathbb{R}^+$ *such that*

$$(D)\sum \left|f(\xi)(v-u) - F(v) + F(u)\right| < \epsilon$$

for any Henstock δ*-fine partial division* $D = \{([u,v],\xi)\}$ *of* $[a,b]$.

Definition 2.3 [2] A sequence $\langle f_n \rangle$ of Henstock integrable functions on $[a,b]$ is *equi-integrable* on $[a,b]$ if for any $\epsilon > 0$, there exists $\delta(\xi) > 0$ such that for each n, we have

$$\left|(D)\sum f_n(\xi)(v-u) - (\mathcal{H})\int_a^b f_n\right| < \epsilon,$$

whenever $D = \{([u,v],\xi)\}$ is a Henstock δ-fine division of $[a,b]$.

Theorem 2.4 *Let $\langle f_n \rangle$ be a sequence of Henstock integrable functions on $[a, b]$. If $\langle f_n \rangle$ is equi-integrable on $[a, b]$ and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for each $x \in [a, b]$, then f is Henstock integrable and

$$\lim_{n\to\infty} (\mathcal{H}) \int_a^b f_n = (\mathcal{H}) \int_a^b f.$$

Definition 2.5 Let $X \subseteq \mathbb{R}$. The function $\mathbf{1}_X : \mathbb{R} \to \mathbb{R}$ define by

$$\mathbf{1}_X(x) = \begin{cases} 1, & \text{if } \ x \in X, \\ 0, & \text{if } \ x \notin X. \end{cases}$$

is called the *characteristic function* on X.

Definition 2.6 Let $X \subseteq [a, b]$. We say that f *is Henstock integrable on* X if the function $f \cdot \mathbf{1}_X$ is Henstock integrable on $[a, b]$ and we write

$$(\mathcal{H}) \int_X f = (\mathcal{H}) \int_a^b (f \cdot \mathbf{1}_X).$$

Definition 2.7 [8] A sequence $\langle f_n \rangle$ of Henstock integrable functions on $[a, b]$ is said to be *uniformly gauge Cauchy* on $[a, b]$ if for any $\epsilon > 0$, there exists a gauge $\delta(\xi) > 0$ and a positive integer N such that for each $n, m \geq N$, we have

$$\left| (D) \sum f_n(\xi)(v - u) - (D) \sum f_m(\xi)(v - u) \right| < \epsilon$$

whenever $D = \{([u, v], \xi)\}$ is a Henstock δ-fine division of $[a, b]$.

Theorem 2.8 [8] *Let $\langle f_n \rangle$ be a sequence of Henstock integrable functions on $[a, b]$. The following are equivalent:*

(*i*) $\langle f_n \rangle$ *is uniformly gauge Cauchy.*

(*ii*) $\left\langle (\mathcal{H}) \int_a^b f_n \right\rangle$ *converges and* $\langle f_n \rangle$ *is equi-integrable on* $[a, b]$.

Definition 2.9 Let $F : [a, b] \to \mathbb{R}$ be a function and $[c, d] \subseteq [a, b]$. The *oscillation of* F *over* $[c, d]$, denoted by $\omega(F; [c, d])$, is given by

$$\omega(F; [c, d]) = \sup \big\{ |F(y) - F(x)| : c \leq x < y \leq d \big\}.$$

Geometrically, the oscillation of a function F over $[c, d]$ is just the "*diameter*" of the image set $F([c, d])$.

3 Results

Here, we give a version of the Harnack Extension and provide an alternative proof of it. First, we prove the following Lemma.

Lemma 3.1 *Let* $f : [a, b] \to \mathbb{R}$ *be a function. Suppose the following conditions hold:*

(*i*) *there exists an increasing sequence* $\langle X_n \rangle$ *of non-empty subsets of* $[a, b]$ *such that* $[a, b] = \bigcup_{n=1}^{\infty} X_n$ *and* f *is Henstock integrable on each* X_n *and*

(*ii*) *for every* $\epsilon > 0$*, there exists a positive integer* N *such that for all* $n \geq N$*, there exists* $\delta_n(\xi) > 0$ *on* $[a, b]$ *such that for all Henstock* δ_n*-fine partial division* $P = \{([u, v], \xi)\}$ *of* $[a, b]$ *with* $\xi \notin X_n$*, we have*

$$\left| (P) \sum f(\xi)(v - u) \right| < \epsilon.$$

Then the sequence $\langle f \cdot \mathbf{1}_{X_n} \rangle$ *is equi-integrable on* $[a,b]$ *and the sequence* $\left\langle (\mathcal{H}) \int_a^b (f \cdot \mathbf{1}_{X_n}) \right\rangle$ *converges.*

Proof: For each $n \in \mathbb{N}$, let $f_n = f \cdot \mathbf{1}_{X_n}$. We will show that $\langle f_n \rangle$ is uniformly gauge Cauchy on $[a,b]$. Now, let $\epsilon > 0$. Then there exists a positive integer N such that for all $n \geq N$, there exists $\delta_n(\xi) > 0$ on $[a,b]$ such that for all Henstock δ_n-fine partial division $P = \{([u,v],\xi)\}$ of $[a,b]$ with $\xi \notin X_n$, we have

$$\left|(P)\sum f(\xi)(v-u)\right| < \frac{\epsilon}{2}.$$

Define $\delta(\xi) = \delta_N(\xi)$ for all $\xi \in [a,b]$. Let $n > m \geq N$ and
$D = \{([u,v],\xi)\}$ be any Henstock δ-fine division of $[a,b]$. Let $P_1 \subseteq D$ and $P_2 \subseteq D$ such that $\xi \in X_N$ and $\xi \notin X_N$, respectively. Because $\langle X_n \rangle$ is increasing, we note that if $\xi \in X_N$, then $f_n(\xi) - f_m(\xi) = 0$; and if $\xi \notin X_N$, then

$$f_n(\xi) - f_m(\xi) = \begin{cases} 0 & , \text{ if } \xi \notin X_n; \\ -f(\xi) & , \text{ if } \xi \in X_m \setminus X_N; \\ f(\xi) & , \text{ if } \xi \in X_n \setminus X_m. \end{cases}$$

Thus,

$$\begin{aligned} &\left|(D)\sum f_n(\xi)(v-u) - (D)\sum f_m(\xi)(v-u)\right| \\ &\qquad = \left|(D)\sum \big[f_n(\xi) - f_m(\xi)\big](v-u)\right| \\ &\qquad \leq \left|(P_1)\sum \big[f_n(\xi) - f_m(\xi)\big](v-u)\right| \\ &\qquad\qquad + \left|(P_2)\sum \big[f_n(\xi) - f_m(\xi)\big](v-u)\right| \end{aligned}$$

$$
\begin{aligned}
&= \left|(P_2)\sum\left[f_n(\xi)-f_m(\xi)\right](v-u)\right| \\
&\le \left|(P_2)\sum_{\xi\in X_m\setminus X_N}\left[-f(\xi)\right](v-u)\right| \\
&\qquad +\left|(P_2)\sum_{\xi\in X_n\setminus X_m} f(\xi)(v-u)\right| \\
&= \left|(P_2)\sum_{\xi\in X_m\setminus X_N} f(\xi)(v-u)\right| \\
&\qquad +\left|(P_2)\sum_{\xi\in X_n\setminus X_m} f(\xi)(v-u)\right| \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}
$$

This shows that $\langle f_n\rangle$ is uniformly gauge Cauchy. Hence, by Theorem 2.8, $\langle f_n\rangle$ is equi-integrable on $[a,b]$ and $\left\langle (\mathcal{H})\int_a^b f_n\right\rangle$ converges. □

Lemma 3.2 *Let $f:[a,b]\to\mathbb{R}$ be a function. Suppose that there exists an increasing sequence $\langle X_n\rangle$ of non-empty subsets of $[a,b]$ such that $[a,b]=\bigcup_{n=1}^{\infty}X_n$ and f is Henstock integrable on each X_n. If for every $\epsilon>0$, there exists a positive integer N such that for all $n\ge N$, there exists $\delta_n(\xi)>0$ on $[a,b]$ such that for all Henstock δ_n-fine partial division $P=\{([u,v],\xi)\}$ of $[a,b]$ with $\xi\notin X_n$, we have*

$$\left|(P)\sum f(\xi)(v-u)\right|<\epsilon,$$

then the sequence $\langle f\cdot\mathbf{1}_{X_n}\rangle$ is equi-integrable on $[a,b]$ and f is Henstock integrable on $[a,b]$. Moreover,

$$(\mathcal{H})\int_a^b f=\lim_{n\to\infty}(\mathcal{H})\int_a^b(f\cdot\mathbf{1}_{X_n}).$$

Proof: For each n, let $f_n = f \cdot \mathbf{1}_{X_n}$. By Lemma 3.1, $\langle f_n \rangle$ is equi-integrable on $[a,b]$ and $\left\langle (\mathcal{H}) \int_a^b f_n \right\rangle$ converges.

Now, let $\epsilon > 0$. For all $x \in [a,b]$, there exists a positive integer $N(x) \in \mathbb{N}$ such that $x \in X_{N(x)}$. Since $\langle X_n \rangle$ is increasing, $x \in X_n$ for all $n \geq N(x)$. Thus, for all $n \geq N(x)$

$$\begin{aligned} \left| f_n(x) - f(x) \right| &= \left| (f \cdot \mathbf{1}_{X_n})(x) - f(x) \right| \\ &= \left| f(x) - f(x) \right| \\ &< \epsilon. \end{aligned}$$

This implies that $f_n \to f$ pointwisely on $[a,b]$. Hence, by Theorem 2.4, f is Henstock integrable on $[a,b]$ and

$$(\mathcal{H}) \int_a^b f = \lim_{n\to\infty} (\mathcal{H}) \int_a^b f_n. \qquad \square$$

We now state and prove the Harnack Extension.

Theorem 3.3 (Harnack Extension) *Let $f : [a,b] \to \mathbb{R}$ and X be a closed subset of $[a,b]$. Suppose the following conditions are satisfied:*

(i) $(a,b) \setminus X = \bigcup_{k=1}^{\infty} (c_k, d_k)$, where $\{(c_k, d_k)\}$ is a collection of pairwise disjoint open intervals and

(ii) f is Henstock integrable on X and on each $[c_k, d_k]$ with

$$\sum_{k=1}^{\infty} \omega(F_k; [c_k, d_k]) < \infty,$$

where F_k denotes the primitive of f on $[c_k, d_k]$.

For each $n \in \mathbb{N}$, let $X_n = X \cup \left(\bigcup_{k=1}^{n} (c_k, d_k) \right)$ and $f_n = f \cdot \mathbf{1}_{X_n}$. Then $\langle f_n \rangle$ is equi-integrable and f is Henstock integrable on $[a,b]$. In this case,

$$(\mathcal{H}) \int_a^b f = \lim_{n\to\infty} (\mathcal{H}) \int_a^b f_n = (\mathcal{H}) \int_X f + \sum_{k=1}^{\infty} F_k(c_k, d_k).$$

Proof: We assume that $a, b \in X$. Then

$$[a,b] = \bigcup_{n=1}^{\infty} X_n = X \cup (c_1,d_1) \cup (c_2,d_2) \cup (c_3,d_3) \cup \cdots$$

and f is Henstock integrable on each X_n. Since X and (c_k, d_k) are pairwise disjoint, we have

$$(\mathcal{H}) \int_a^b f_n = (\mathcal{H}) \int_X f + \sum_{k=1}^{n} F_k(c_k,d_k) \text{ , for all } n.$$

Let $\epsilon > 0$. By Henstock Lemma (Theorem 2.2), there exists $\delta_k(\xi) > 0$ on $[a,b]$ such that whenever $D = \{([u,v],\xi)\}$ is a Henstock δ_k-fine division of $[a,b]$,we have

$$(D) \sum \left| (f \cdot \mathbf{1}_{[c_k,d_k]})(\xi)(v-u) - F_k(u,v) \right| < \frac{\epsilon}{2^{k+1}} \text{ , for each } k.$$

Since each (c_k, d_k) is open, we may assume that

$$(\xi - \delta_k(\xi), \xi + \delta_k(\xi)) \subseteq (c_k, d_k)$$

if $\xi \in (c_k, d_k)$. Since $\displaystyle\sum_{k=1}^{\infty} \omega(F_k; [c_k,d_k]) < \infty$, there exists a positive integer N such that

$$\sum_{k=N}^{\infty} \omega(F_k; [c_k,d_k]) < \frac{\epsilon}{2}.$$

Note that because X and (c_k, d_k) are pairwise disjoint, we have

$$\xi \notin X_n = X \cup (c_1,d_1) \cup (c_2,d_2) \cup \cdots \cup (c_n,d_n)$$

$$\iff \quad \xi \in \bigcup_{k=n+1}^{\infty} (c_k,d_k).$$

Thus, for all $n \geq N$ and for any Henstock δ_n-fine partial division $P = \{([u,v],\xi)\}$ of $[a,b]$ with $\xi \notin X_n$, we have

$$\begin{aligned}
&\left|(P)\sum f(\xi)(v-u)\right| \\
=\ & \left|(P)\sum_{\xi\in\bigcup_{k=n+1}^{\infty}(c_k,d_k)} f(\xi)(v-u)\right| \\
\leq\ & \sum_{k=n+1}^{\infty}\left[(P)\sum_{\xi\in(c_k,d_k)}\left|(f\cdot\mathbf{1}_{[c_k,d_k]})(\xi)(v-u)-F_k(u,v)\right|\right] \\
&\quad+\sum_{k=n+1}^{\infty}\left[(P)\sum_{\xi\in(c_k,d_k)}|F_k(u,v)|\right] \\
<\ & \sum_{k=1}^{\infty}\frac{\epsilon}{2^{k+1}}+\sum_{k=N}^{\infty}\omega(F_k;[c_k,d_k]) \\
<\ & \frac{\epsilon}{2}+\frac{\epsilon}{2} \\
=\ & \epsilon.
\end{aligned}$$

By Lemma 3.2, then the sequence $\langle f_n\rangle$ is equi-integrable on $[a,b]$ and f is Henstock integrable on $[a,b]$. Moreover,

$$(\mathcal{H})\int_a^b f=\lim_{n\to\infty}(\mathcal{H})\int_a^b f_n=(\mathcal{H})\int_X f+\sum_{k=1}^{\infty}F_k(c_k,d_k).\ \square$$

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