The Mindanawan Journal of Mathematics
Official Journal of the Department of Mathematics and Statistics
Mindanao State University-Iligan Institute of Technology
ISSN: 2094-7380 (Print) | 2783-0136 (Online)

Vol. 4 (2022), no. 2, pp. 35–45

θ-Preopen Sets on Topological Spaces

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Received: 27th October 2022 Revised: 5th December 2022

Abstract

In this paper, we revisit the concept of θ-preopen set defined by Noiri [17], and then investigate the connection of this set to the other well-known concepts in topology such as the classical open, θ-open, and preopen sets. We also investigate the concept of θ-precontinuous and strongly θ-precontinuous functions from an arbitrary topological space into the product space.

1 Introduction and Preliminaries

The first initiation to develop different versions of open sets was done by Levine [13] in 1963 where he introduce the concepts of semi-open set, semi-closed set and semi-continuity of a function.

A subset O of a topological space X is semi-open [13] if $O \subseteq Cl(Int(O))$. Equivalently, O is semi-open if there exists an open set G in X such that $G \subseteq O \subseteq Cl(G)$. A subset F of X is semi-closed if its complement $X \setminus F$ is semi-open in X. Let A be a subset of a space X. A point $p \in X$ is a semi-closure point of A if for every semi-open set G in X containing x, $G \cap A \neq \varnothing$. We denote by $sCl(A)$ the set of all semi-closure points of A.

In 1968, Veličko [22] introduce the concept θ-continuity between topological spaces and defined the concepts of θ-closure and θ-interior of a set. The work of Veličko was pursued by Dickman and Porter [4, 5], Joseph [10], and Long and Herrington [14]. Numerous authors then have obtained interesting results related to θ-open sets, see [1, 3, 8, 9, 11, 20].

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The θ-closure and θ-interior of A are, respectively, defined by

$$Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \varnothing \text{ for every open set } U \text{ containing } x\}$$

and $Int_\theta(A) = \{x \in X : Cl(U) \subseteq A \text{ for some open set } U \text{ containing } x\}$, where $Cl(U)$ is the closure of U in X. A is θ-closed [22] if $Cl_\theta(A) = A$ and θ-open [22] if $Int_\theta(A) = A$. Equivalently, A is θ-open if and only if $X \setminus A$ is θ-closed. It is known that the collection $\mathcal{T}_\theta$ of all θ-open sets forms a topology on X, which is strictly coarser that $\mathcal{T}$.

2020 Mathematics Subject Classification: 54A10, 54A05
Keywords and Phrases: θ-preopen set, θ-preopen function, θ-precontinuous function, strongly θ-precontinuous function, θ-preconnected space
-This research has been supported by the Mindanao State University-Iligan Institute of Technology through the Office of the Vice Chancellor for Research and Extension under the 2021 Research Program in Graph Theory, Algebra, and Analysis (2021 ResProGraThAA).

In 1980, Mashhour et al. [15] introduced the concepts of preopen and weak preopen sets, precontinuous and weak precontinuous funnctions, and preopen and weak preopen functions on topological spaces.

A set $A \subseteq X$ is said to be preopen [16] if $O \subseteq Int(Cl(O))$. A subset F of X is called preclosed if the complement of F is preopen. The preclosure of A, denoted by $pCl(A)$, is the intersection of all preclosed sets containing A and the preinterior of A, denoted by $pInt(A)$, is the union of all preopen sets contained in A.

Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\mathcal{T}_\alpha$ be the topology on Y_α. The Tychonoff topology on $\Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $p_\alpha^{-1}(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of $\mathcal{T}_\alpha$, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \dots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}, U_{\alpha_2} \subseteq Y_{\alpha_2}, \dots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \Pi_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, ..., B_{\alpha_n} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in \{1, 2, \dots, n\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in \mathcal{A}$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α-th coordinate projection map.

In this paper, the concept of θ-preopen set defined by Noiri [17] will be revisited and investigated further. Some topological concepts related to θ-preopen sets will also be defined and studied.

2 θ-Preopen and θ-Preclosed Functions

In this section, we define and characterize the concepts of θ-preopen and θ-preclosed functions.

Definition 2.1. Let X be a topological space and $A \subseteq X$. A point $x \in X$ is called a θ-precluster (or pre θ-cluster [17, p.286]) point of A if $pCl(U) \cap A \neq \varnothing$ for every preopen set U containing x. The set of all θ-precluster points of A is called the θ-preclosure (or pre θ-closure [17, p.286]) of A, and is denoted by $pCl_\theta(A)$. A subset A of X is said to be θ-preclosed (or pre θ-closed [21]) if $A = pCl_\theta(A)$. A subset A of X is said to be θ-preopen (or pre θ-open [17, p.286]) if its complement $X \setminus A$ is θ-preclosed.

The next result characterizes that concept of a θ-preopen set.

Lemma 2.2. *Let X be a topological space. $A \subseteq X$ is θ-preopen if and only if for every $x \in A$, there exists a preopen set U containing x such that $pCl(U) \subseteq A$.*

Proof. Suppose that A is θ-preopen. Let $x \in A$. Then $X \setminus A = pCl_\theta(X \setminus A)$, so that x is not a θ-precluster point of $X \setminus A$. This means that for some preopen set U containing x, $pCl(U) \cap X \setminus A = \varnothing$, or equivalently, $pCl(U) \subseteq A$.

To verify the converse, let $x \in pCl_\theta(X \setminus A)$. Suppose further that $x \notin X \setminus A$, that is, $x \in A$. By assumption, there exists a preopen set U containing x such that $pCl(U) \subseteq A$, or equivalently, $pCl(U) \cap X \setminus A = \varnothing$. This means that x is not a θ-precluster point of $X \setminus A$. Hence, $x \notin pCl_\theta(X \setminus A)$, a contradiction. Thus, $pCl_\theta(X \setminus A) = X \setminus A$, that is, A is θ-preopen. □

Theorem 2.3. *Let X be a topological space.*

(i) *If $A \subseteq X$ is θ-open, then A is θ-preopen but not conversely.*

(ii) *If $A \subseteq X$ is θ-preopen, then A is preopen but not conversely.*

(iii) *θ-preopen and open sets are two independent notions.*

Proof. (i): Suppose that A is θ-open. Let $x \in A$. Then there exists an open set U containing x such that $Cl(U) \subseteq A$. Since U is open, $U = Int(U)$ and U is preopen [18, p.1009]. By [2, Theorem 1.5],

$$pCl(U) = U \cup Cl(Int(U)) = U \cup Cl(U) = Cl(U) \subseteq A.$$

In view of Lemma 2.2, A is θ-preopen.

To show that the converse is not necessarily true, consider $X = \{a, b, c, d\}$ with topology $\mathcal{T}_1 = \{\varnothing, X, \{a, b\}, \{c, d\}\}$. Then $\{b, c\}$ is θ-preopen but not θ-open.

(ii): Suppose that A is θ-preopen. Let $x \in A$. Then there exists a preopen set U containing x such that $U \subseteq pCl(U) \subseteq A$. Since U is preopen, $U \subseteq Int(Cl(U)) \subseteq Int(Cl(A)) \subseteq Cl(A)$. Let $O := Int(Cl(U))$, which is an open set containing x and $O \subseteq Cl(A)$. It follows that $x \in Int(Cl(A))$. Accordingly, A is preopen.

To show that the converse is not necessarily true, consider $X = \{a, b, c, d\}$ with topology $\mathcal{T}_2 = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Then $\{a, b, c\}$ preopen but not θ-preopen.

(iii): From $(X, \mathcal{T}_2)$, $\{a, c\}$ is open but not θ-preopen and from $(X, \mathcal{T}_1)$, $\{b, c\}$ is θ-preopen but not open. □

Remark 2.4. The following diagram holds for a subset of a topological space.

$$\begin{array}{ccc} \theta\text{-preopen} & \longrightarrow & \text{preopen} \\ \uparrow & & \uparrow \\ \theta\text{-open} & \longrightarrow & \text{open} \end{array}$$

We also remark that the above diagram is also true for their respective closed sets. The reverse implications are not true as shown in Theorem 2.3.

Definition 2.5. Let X be a topological space and $A \subseteq X$. The θ-preinterior of A is denoted and defined by $pInt_\theta(A)=\cup\{U : U$ is θ-preopen and $U \subseteq A\}$.

Remark 2.6. The arbitrary union of θ-preopen sets is θ-preopen.

We note that by Remark 2.6, $pInt_\theta(A)$ is the largest θ-preopen set contained in A. Moreover, $x \in pInt_\theta(A)$ if and only if there exists a θ-preopen set U containing x such that $U \subseteq A$.

Remark 2.7. Let X be a topological space and $A, B \subseteq X$. Then

(i) If $A \subseteq B$, then $pCl_\theta(A) \subseteq pCl_\theta(B)$.

(ii) $pCl_\theta(A)=\cap\{F : F$ is θ-preclosed and $A \subseteq F\}$.

(iii) $pCl_\theta(A)$ is the smallest θ-preclosed set containing A.

(iv) $x \in pCl_\theta(A)$ if and only if for every θ-preopen set U containing x, $U \cap A \neq \varnothing$.

(v) $pCl_\theta(pCl_\theta(A)) = pCl_\theta(A)$.

(vi) $pCl_\theta(A \cup B) = pCl_\theta(A) \cup pCl_\theta(B)$.

(vii) $A \subseteq pCl_\theta(A) \subseteq Cl_\theta(A)$.

(viii) If $A \subseteq B$ then $pInt_\theta(A) \subseteq pInt_\theta(B)$.

(ix) A is θ-preopen if and only if $A = pInt_\theta(A)$.

(x) $pInt_\theta(A) = pInt_\theta(pInt_\theta(A))$.

(xi) $pInt_\theta(A \cap B) = pInt_\theta(A) \cap pInt_\theta(B)$.

(xii) $x \in pInt_\theta(A)$ if and only if there exists a preopen set U containing x such that $pCl(U) \subseteq A$.

(xiii) $Int_\theta(A) \subseteq pInt_\theta(A) \subseteq A$.

(xiv) $pCl_\theta(X \setminus A) = X \setminus pInt_\theta(A)$.

(xv) $pInt_\theta(X \setminus A) = X \setminus pCl_\theta(A)$.

Next, we characterize the concepts of θ-preopen, θ-preclosed, strongly θ-preopen, and strongly θ-preclosed functions.

Definition 2.8. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be θ-preopen (resp., θ-preclosed, strongly θ-preopen, strongly θ-preclosed) on X if $f(G)$ is θ-preopen (resp., θ-preclosed, θ-preopen, θ-preoclosed) in Y for every θ-open (resp., θ-closed, open, closed) set G in X.

Remark 2.9. Every strongly θ-preopen (resp., strongly θ-preclosed) function is θ-preopen (resp., θ-preclosed) but not conversely.

We note that θ-preopen and θ-preclosed (strongly θ-preopen and strongly θ-preclosed) functions are equivalent if f is bijective.

Theorem 2.10. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ-preopen on X.*

(ii) *$f(Int_\theta(A)) \subseteq pInt_\theta(f(A))$ for every $A \subseteq X$.*

(iii) *For each $x \in X$ and for every open set U in X containing x, there exists a preopen set V in Y containing $f(x)$ such that $pCl(V) \subseteq f(U)$.*

Proof. (i)⇒(ii): Suppose that f is θ-preopen on X. Let $A \subseteq X$. Then $f(Int_\theta(A)) \subseteq f(A)$. Since $Int_\theta(A)$ is θ-open, $f(Int_\theta(A))$ is θ-preopen in Y contained in $f(A)$. Since $pInt_\theta(f(A))$ is the largest θ-preopen set contained in $f(A)$, $f(Int_\theta(A)) \subseteq pInt_\theta(f(A))$.

(ii)⇒(iii): Let $x \in X$ and U be open in X containing x. Note that $Cl(U) \cap (X \setminus Cl(U)) = \varnothing$. This means that $x \notin Cl_\theta((X \setminus Cl(U))) = X \setminus Int_\theta(Cl(U)) \subseteq X \setminus Int_\theta(U)$. It follows that $f(x) \in f(Int_\theta(U)) \subseteq pInt_\theta(f(U))$. In view of Remark 2.7 (xii), there exists a preopen set V containing $f(x)$ such that $pCl(V) \subseteq f(U)$.

(iii)⇒(iv): Let U be θ-open in X. Let $y \in f(U)$. Then there exists $x \in X$ such that $f(x) = y$. By assumption, there exists a preopen set V containing y such that $pCl(V) \subseteq f(U)$. By Lemma 2.2, $f(U)$ is θ-preopen. □

Theorem 2.11. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ-preclosed on X.*

(ii) *$pCl_\theta(f(B)) \subseteq f(Cl_\theta(B))$ for each $B \subseteq X$.*

Proof. (i)$\Rightarrow$(ii): Suppose that f is θ-preclosed on X. Let $B \subseteq X$. Then $f(B) \subseteq f(Cl_\theta(B))$. Since $Cl_\theta(B)$ is θ-closed, $f(Cl_\theta(B))$ is θ-preclosed in Y containing $f(B)$. Since $pCl_\theta(f(B))$ is the smallest θ-preclosed set containing in $f(B)$, $pCl_\theta(f(B)) \subseteq f(Cl_\theta(B))$.

(ii)$\Rightarrow$(iii): Let U be θ-closed in X. Then $U = Cl_\theta(U)$. By assumption, $f(U) \subseteq pCl_\theta(f(U)) \subseteq f(Cl_\theta(U)) = f(U)$. Thus, $f(U)$ is θ-preclosed. □

Following the same argument as in Theorems 2.10 and 2.11, respectively, the following two results hold.

Theorem 2.12. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is strongly θ-preopen on X.*

(ii) *$f(Int(A)) \subseteq pInt_\theta(f(A))$ for every $A \subseteq X$.*

(iii) *$f(B)$ is θ-preopen for every basic open set B in X.*

(iv) *For each $x \in X$ and for every open set U in X containing x, there exists an open set V in Y containing $f(x)$ such that $pCl(V) \subseteq f(U)$.*

Theorem 2.13. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is strongly θ-preclosed on X.*

(ii) *$pCl_\theta(f(G)) \subseteq f(Cl(G))$ for each $G \subseteq X$.*

3 Strongly θ-Precontinuous Functions in the Product Space

This section provides a characterization of a θ-precontinuous function [17, p.286] and strongly θ-precontinuous function [19, p.308] from an arbitrary topological space into the product space.

Definition 3.1. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be θ-precontinuous [17, p.286] (resp., strongly θ-precontinuous [19, p.308]) on X if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a preopen set U containing x such that $f(pCl(U)) \subseteq Cl(V)$ (resp., $f(pCl(U)) \subseteq V$).

Theorem 3.2. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ-precontinuous on X.*

(ii) *$f(pCl_\theta(A) \subseteq Cl_\theta(f(A))$ for each $A \subseteq X$.*

(iii) *$pCl_\theta(f^{-1}((B)) \subseteq f^{-1}(Cl_\theta(B))$ for each $B \subseteq Y$.*

(iv) *$f^{-1}(G)$ is θ-preopen in X for each θ-open subset G of Y.*

(v) *$f^{-1}(F)$ is θ-preclosed in X for each θ-closed subset F of Y.*

Proof. The equivalence of (i), (ii), (iii) is proved in [17, Theorem 3.1].

(iii)⇒(v): Let F be θ-closed in Y. Then $Cl_\theta(F) = F$ so that

$$pCl_\theta(f^{-1}(F)) \subseteq f^{-1}(Cl_\theta(F)) = f^{-1}(F) \subseteq pCl_\theta(f^{-1}(F)).$$

This means that $f^{-1}(F)$ is θ-preclosed in X.

(v)⇒(iii): Let $B \subseteq Y$. In view of [12, Lemma 2], $Cl_\theta(B)$ is θ-closed. By assumption, $f^{-1}(Cl_\theta(B))$ is θ-preclosed containing $f^{-1}(B)$. Using Remark 2.7 (iii), $pCl_\theta(f^{-1}((B)) \subseteq f^{-1}(Cl_\theta(B))$.

(iii)⇔(v): This is immediate since the inverse mapping preserves set operations. □

Following the same argument as in Theorems 3.2, the following result holds.

Theorem 3.3. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is strongly θ-precontinuous on X.*

(ii) *$f^{-1}(G)$ is θ-preopen in X for each open subset G of Y.*

(ii) *$f^{-1}(F)$ is θ-preclosed in X for each closed subset F of Y.*

(iii) *$f^{-1}(B)$ is θ-preopen for each (subbasic) basic open set B in Y.*

(iv) *$f(pCl_\theta(A) \subseteq Cl(f(A))$ for each $A \subseteq X$.*

(v) *$pCl_\theta(f^{-1}((B)) \subseteq f^{-1}(Cl(B))$ for each $B \subseteq Y$.*

The proof of the following result is standard, hence omitted.

Theorem 3.4. *Let X and Y be topological spaces and $f_A : X \to \mathcal{D}$ the characteristic function of subset A of X, where $\mathcal{D}$ is the set $\{0, 1\}$ with discrete topology. Then f_A is θ-precontinuous if and only if f_A is strongly θ-precontinuous if and only if A is both θ-preopen and θ-preclosed.*

In the following results, if $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is a product space and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, we denote $A_{\alpha_1} \times \cdots \times A_{\alpha_n} \times \Pi\{Y_\alpha : \alpha \notin K\}$ by $\langle A_{\alpha_1}, \ldots, A_{\alpha_n}\rangle$, where $K = \{\alpha_1, \ldots, \alpha_n\}$.

If $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ is a finite product, denote $A_1 \times \cdots \times A_n$ by $\langle A_1, \ldots, A_n\rangle$.

Theorem 3.5. *Let X be a topological space and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space. A function $f : X \to Y$ is strongly θ-precontinuous on X if and only if $p_\alpha \circ f$ is strongly θ-precontinuous on X for every $\alpha \in \mathcal{A}$.*

Proof. Assume that f is strongly θ-precontinuous on X. Let $\alpha \in \mathcal{A}$ and O_α be open in Y_α. Since p_α is continuous, $p_\alpha^{-1}(O_\alpha)$ is open in Y. Hence, $f^{-1}(p_\alpha^{-1}(O_\alpha)) = (p_\alpha \circ f)^{-1}(O_\alpha)$ is θ-preopen in X. Thus, $p_\alpha \circ f$ is strongly θ-precontinuous for every $\alpha \in \mathcal{A}$.

Conversely, assume that each coordinate function $p_\alpha \circ f$ is strongly θ-precontinuous. Let G_α be open in Y_α. Then $\langle G_\alpha\rangle$ is a subbasic open set in Y and $(p_\alpha \circ f)^{-1}(G_\alpha) = f^{-1}(p_\alpha^{-1}(G_\alpha)) = f^{-1}(\langle G_\alpha\rangle)$ is θ-preopen in X. In view of Theorem 3.3 (iii), f is strongly θ-precontinuous on X. □

Corollary 3.6. *Let X be a topological space, $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space, and $f_\alpha : X \to Y_\alpha$ be a function for each $\alpha \in \mathcal{A}$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x)\rangle$. Then f is strongly θ-precontinuous on X if and only if each f_α is strongly θ-precontinuous on X for each $\alpha \in \mathcal{A}$.*

Lemma 3.7. [6, 17] *Let $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space, $K := \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq \mathcal{A}$ and $\varnothing \neq U_{\alpha_i} \subseteq Y_{\alpha_i}$ for each $\alpha_i \in K$. Then*

(i) *$U = \langle U_{\alpha_1}, \dots, U_{\alpha_n}\rangle$ is preopen in Y if and only if each U_{α_i} is preopen in Y_{α_i}; and*

(ii) *$pCl(\Pi\{A_\alpha : \alpha \in \mathcal{A}\}) \subseteq \Pi\{pCl(A_\alpha) : \alpha \in \mathcal{A}\}$.*

Theorem 3.8. *Let $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. Then $pCl_\theta(\Pi\{A_\alpha : \alpha \in \mathcal{A}\}) \subseteq \Pi\{pCl_\theta(A_\alpha) : \alpha \in \mathcal{A}\}$.*

Proof. Let $x = \langle x_\alpha\rangle \in pCl_\theta(\Pi\{A_\alpha : \alpha \in \mathcal{A}\})$. Suppose that $x \notin \Pi\{pCl_\theta(A_\alpha) : \alpha \in \mathcal{A}\}$. Then $x_\beta \notin pCl_\theta(A_\beta)$ for some $\beta \in \mathcal{A}$. This means that there exists a preopen set $U_\beta \ni x_\beta$ such that $pCl_\theta(U_\beta) \cap A_\beta = \varnothing$. In view of Lemma 3.7, $\langle U_\beta\rangle$ is preopen, $x \in \langle U_\beta\rangle$, and

$$\begin{aligned} pCl(\langle U_\beta\rangle) \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} &\subseteq \langle pCl(U_\beta)\rangle \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} \\ &= \Pi\{A_\alpha : \alpha \neq \beta\} \times (pCl(U_\beta) \cap A_\beta) \\ &= \varnothing. \end{aligned}$$

This is a contradiction to the assumption that $x \in pCl_\theta(\Pi\{A_\alpha : \alpha \in \mathcal{A}\})$. Thus, $x \in \Pi\{pCl_\theta(A_\alpha) : \alpha \in \mathcal{A}\}$. □

Theorem 3.9. *Let $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i = 1, \dots, n$. Then $\langle pInt_\theta(A_1), \dots, pInt_\theta(A_n)\rangle \subseteq pInt_\theta(\langle A_1, \dots, A_n\rangle)$.*

Proof. Let $x = \langle x_i\rangle \in \langle pInt_\theta(A_1), \dots, pInt_\theta(A_n)\rangle$. Then $x_i \in pInt_\theta(A_i)$ for all $i = 1, 2, \dots, n$. This means that there exists a preopen set $U_i \ni x_i$ such that $pCl(U_i) \subseteq A_i$. In view of Lemma 3.7, $\langle U_1, U_2, \dots, U_n\rangle$ is preopen containing x and

$$pCl(\langle U_1, \dots, U_n\rangle) \subseteq \langle pCl(U_1), \dots, pCl(U_n)\rangle \subseteq \langle A_1, \dots, A_n\rangle.$$

Hence, $x \in pInt_\theta(\langle A_1, \dots, A_n\rangle)$. □

Theorem 3.10. *Let $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space and $\varnothing \neq O_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. Then $O = \langle O_{\alpha_1}, \dots, O_{\alpha_n}\rangle$ is θ-preopen in Y if each O_{α_i} is θ-preopen in Y_{α_i}.*

Proof. Suppose that each $O_{\alpha_i} \neq \varnothing$ is θ-preopen in Y_{α_i}. Let $x = \langle x_\alpha\rangle \in O$. Then $x_{\alpha_i} \in O_{\alpha_i}$ for all $\alpha_i \in K$. Hence, there exists a preopen set $U_{\alpha_i} \ni x_{\alpha_i}$ such that $pCl(U_{\alpha_i}) \subseteq O_{\alpha_i}$. Let $U = \langle U_{\alpha_1}, \dots, U_{\alpha_n}\rangle$. By Lemma 3.7, U is preopen containing x and

$$\begin{aligned} pCl(U) &= pCl(\langle U_{\alpha_1}, \dots, U_{\alpha_n}\rangle) \\ &\subseteq \langle pCl(U_{\alpha_1}), \dots, pCl(U_{\alpha_n})\rangle \\ &\subseteq \langle O_{\alpha_1}, \dots, O_{\alpha_n}\rangle \\ &= O. \end{aligned}$$

Thus, O is a θ-preopen set in Y. □

Theorem 3.11. [17, Theorem 4.7] *Let $X = \Pi\{X_\alpha : \alpha \in \mathcal{A}\}$ and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be product spaces and for each $\alpha \in \mathcal{A}$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function. If each f_α is θ-precontinuous on X_α, then the function $f : X \to Y$ defined by $f(\langle x_\alpha\rangle) = \langle f_\alpha(x_\alpha)\rangle$ is θ-precontinuous on X.*

The following result can be proved using the same technique employed by the author in [17, Theorem 4.7]. However, we will give an alternative proof for the following theorem.

Theorem 3.12. *Let $X = \Pi\{X_\alpha : \alpha \in \mathcal{A}\}$ and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be product spaces and for each $\alpha \in \mathcal{A}$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function. If each f_α is strongly θ-precontinuous on X_α, then the function $f : X \to Y$ defined by $f(\langle x_\alpha \rangle) = \langle f_\alpha(x_\alpha) \rangle$ is strongly θ-precontinuous on X.*

Proof. Let $\langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle$ be a basic open set in Y. Then

$$f^{-1}(\langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle) = \left\langle f_{\alpha_1}^{-1}(V_{\alpha_1}), \ldots, f_{\alpha_n}^{-1}(V_{\alpha_n}) \right\rangle.$$

Since each f_{α_i} is strongly θ-precontinuous, by Theorem 3.3 (iii), $f_{\alpha_i}^{-1}(V_{\alpha_i})$ is θ-preopen in X_{α_i}. Let $x = \langle x_\alpha \rangle \in f^{-1}(\langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle)$. Then $x_{\alpha_i} \in f_i^{-1}(V_{\alpha_i})$ for all $\alpha_i \in K$. This means that there exists a preopen set $O_{\alpha_i} \ni x_{\alpha_i}$ such that $pCl(O_{\alpha_i}) \subseteq f_{\alpha_i}^{-1}(V_{\alpha_i})$. By [17, Lemma 4.6], $\langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle$ is preopen in X containing x and

$$\begin{aligned} pCl(\langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle) &\subseteq \langle pCl(O_{\alpha_1}), \ldots, pCl(O_{\alpha_n}) \rangle \\ &\subseteq \left\langle f_{\alpha_1}^{-1}(V_{\alpha_1}), \ldots, f_{\alpha_n}^{-1}(V_{\alpha_n}) \right\rangle \\ &= f^{-1}(\langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle). \end{aligned}$$

This implies that $f^{-1}(\langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle)$ is θ-preopen in X. Thus, f is strongly θ-precontinuous on X. □

4 θ-Preconnected Space and Versions of Separation Axioms

This section characterizes the concepts of θ-preconnected space and some versions of separation axioms.

Definition 4.1. A topological space X is said to be a θ-preconnected (resp., θ-connected [23], connected) if it is not the union of two nonempty disjoint θ-preopen (resp., θ-open, open) sets. Otherwise, X is θ-predisconnected (resp., θ-disconnected [23], disconnected). A subset B of X is θ-preconnected (resp., θ-connected [23], connected) if it is θ-preconnected (resp., θ-connected, connected) as a subspace of X.

In view of Theorem 3.4, the following result holds.

Theorem 4.2. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ-preconnected.*

(ii) *The only subsets of X that are both θ-preopen and θ-preclosed are $\varnothing$ and X.*

(iii) *No θ-precontinuous function $f : X \to \mathcal{D}$ is surjective.*

(iii) *No strongly θ-precontinuous function $f : X \to \mathcal{D}$ is surjective.*

By Remark 2.4, we have the following result.

Theorem 4.3. *If a topological space X is θ-preconnected, then X is θ-connected.*

Since connected and θ-connected spaces [8] are equivalent, the following result holds.

Corollary 4.4. *If a topological space X is θ-preconnected, then X is connected.*

Remark 4.5. The following diagram holds for a subset of a topological space.

$$\theta\text{-preconnected} \longrightarrow \theta\text{-connected} \longleftrightarrow \text{connected}$$

Definition 4.6. A topological space X is said to be

(i) θ-preHausdorff if given any pair of distinct points p, q in X there exist disjoint θ-preopen sets U and V such that $p \in U$ and $q \in V$;

(ii) θ-preregular if for each closed set F and each point $x \notin F$, there exist disjoint θ-preopen sets U and V such that $x \in U$ and $F \subseteq V$;

(iii) θ-prenormal if for every pair of disjoint closed sets E and F of X, there exist disjoint θ-preopen sets U and V such that $E \subseteq U$ and $F \subseteq V$.

The following three results can be proved using the same techniques employed in [7].

Theorem 4.7. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ-preHausdorff.*

(ii) *Let $x \in X$. For $y \neq x$, there exists a θ-preopen set U containing x such that $y \notin pCl_\theta(U)$.*

(iii) *For each $x \in X$, $C = \cap\{pCl_\theta(U) : U$ is θ-preopen containing $x\}=\{x\}$.*

Theorem 4.8. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ-preregular.*

(ii) *For each $x \in X$ and an open set U containing x, there exists θ-preopen set V such that $x \in V \subseteq pCl_\theta(V) \subseteq U$.*

(iii) *For each $x \in X$ and closed set F with $x \notin F$, there exists a θ-preopen set V containing x such that $F \cap pCl_\theta(V) = \varnothing$.*

Theorem 4.9. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ-prenormal.*

(ii) *For each closed set A and an open set $U \supseteq A$, there exists a θ-preopen set V containing A such that $pCl_\theta(V) \subseteq U$.*

(iii) *For each pair of disjoint closed sets A and B, there exists a θ-preopen set V containing A such that $pCl_\theta(V) \cap B = \varnothing$.*

A topological space X is said to be a T_1-space if for each $p, q \in X$ with $p \neq q$, there exist open sets U and V such that $p \in U$, $q \notin U$ and $q \in V$, $p \notin V$.

Theorem 4.10. *Let X be a T_1-space. Then*

(i) *If X is θ-preregular, then X is θ-preHausdorff.*

(ii) *If X is θ-prenormal, then X is θ-preregular.*

Proof. (i): Suppose that X is θ-preregular. Since X is a T_1-space, for each $x, y \in X$ with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$. This implies that $x \notin X \setminus U$ and $y \notin X \setminus V$. Since X is θ-preregular, there exist disjoint θ-preopen sets A and B such that $x \in A$ and $X \setminus U \subseteq B$. Note that $y \in X \setminus U$. Hence, $y \in B$. Thus, X is θ-preHausdorff.

We can prove (ii) by following the same argument used in (i). □

Acknowledgement

The author is grateful to the anonymous referee for reviewing the paper.

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