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Another Look at Geodetic and Hull Numbers of a Graph

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Abstract: Given a connected graph G and two vertices u and v in G, $I_G[u,v]$ is the set consisting of u, v and all vertices lying on some $u-v$ geodesic of G. A subset S of $V(G)$ is called a geodetic set of G if $I_G[S] = V(G)$, where $I_G[S] = \cup_{u,v\in S} I_G[u,v]$. The geodetic number of G, denoted by $g(G)$ is the smallest cardinality of a geodetic set of G. In this paper, we give the geodetic number of the composition of a complete graph K_n and a (connected) graph G.

Keywords: geodesic, geodetic number, path absorbing, hull number, composition

1 Introduction

Let G be a connected simple graph and $u, v \in V(G)$, where $V(G)$ is the vertex set of G. The *distance* $d_G(u,v)$ between u and v in G is the length of a shortest path $P(u,v)$ in G. Any $u-v$ path of length $d_G(u,v)$ is called a $u-v$ *geodesic*. For any two vertices u and v of G, the set $I_G[u,v]$ is the closed interval consisting of u, v and all vertices lying on

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some $u-v$ geodesic of G. For any subset S of $V(G)$, the *closure of* S is $I_G[S] = \bigcup_{u,v \in S} I_G[u,v]$.

A subset $S \subseteq V(G)$ is called a *geodetic set* if $I_G[S] = V(G)$. The *geodetic number* $g(G)$ of G is the minimum cardinality of a geodetic set of G. Any subset S of $V(G)$ of cardinality equal to $g(G)$ is called a *geodetic basis* of G. These concepts were introduced in [9]. It was further investigated in [2], [2], [4], [5], [6] and [7].

A vertex in a graph G is an *extreme vertex* if the subgraph induced by its neighbors is complete. The set of extreme vertices in G is denoted by $Ex(G)$. For convention, we set $Ex(G) = V(K_1)$ if $G = K_1$. For other graph theoretic terms which are assumed here, readers are advised to see [8].

In [3], the authors determined the geodetic number of the composition $H[K_n]$ of a connected graph G and the complete graph K_n of order n.

2 The Geodetic Number of the Composition $K_n[G]$

The *composition* of two graphs H and G, denoted by $H[G]$, is the graph with $V(H[G]) = V(H) \times V(G)$ and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1v_1 \in E(H)$ or $u_1 = v_1$ and $u_2v_2 \in E(G)$.

Theorem 2.1 *Let G be a non-complete graph and K_n the complete graph of order $n \geq 2$. Then $2 \leq g(K_n[G]) \leq 4$.*

Proof: Let G be a non-complete graph and K_n the complete graph of order $n \geq 2$. Since $K_n[G]$ is a non-trivial graph, $g(K_n[G]) \geq 2$. Pick distinct vertices $a, b \in V(K_n)$ and vertices $x, y \in V(G)$ such that $xy \notin E(G)$. Let $S =$

$\{(a,x),(a,y),(b,x),(b,y)\}$ and let $(c,z) \in V(K_n[G]) \setminus S$. Clearly, $c \neq a$ or $c \neq b$. Assume that $c \neq a$. Then $[(a,x),(c,z),(a,y)]$ is an (a,x)-(a,y) geodesic. This implies that $(c,z) \in I_{K_n[G]}[(a,x),(a,y)] \subseteq I_{K_n[G]}[S]$. Therefore, $V(K_n[G]) \subseteq I_{K_n[G]}[S]$; hence S is a geodetic set of $K_n[G]$. By definition, $g(K_n[G]) \leq |S| = 4$. This proves the theorem. □

We will now characterize all non-complete graphs G such that $g(K_n[G]) = 2$.

Theorem 2.2 *Let G be a non-complete graph and K_n the complete graph of order $n \geq 2$. Then $g(K_n[G]) = 2$ if and only if $G = \overline{K}_2$ or G is connected with $g(G) = diam(G) = 2$.*

Proof: Suppose $g(K_n[G]) = 2$, say $S = \{(a,x),(b,y)\}$ is a geodetic basis of $K_n[G]$. Note that since $(K_n[G]) \neq K_2$, $a = b$ and $xy \notin E(G)$. First, suppose that G is disconnected. Suppose further that G has more than two components, say $G_1, G_2, \cdots, G_k$ are the components of G, where $k \geq 3$. If $x \in V(G_i)$ and $y \in V(G_j)$ (i may possibly be equal to j), then $(a,z) \notin I_{K_n[G]}[(a,x),(a,y)]$ for all $z \in V(G_t)$, where $t \neq i,j$. This clearly contradicts the fact that S is a geodetic set of $K_n[G]$. Hence, G has only two components. Furthermore, by a similar argument as above, each of these two componets is the trivial graph. Thus, $G = \overline{K}_2$.

Next, suppose that G is connected and let $S_G = \{x,y\}$. Let $z \in V(G) \setminus S_G$. Then (a,z) is in the interval $I_{K_n[G]}[(a,x),(a,y)]$. Since $d_{K_n[G]}((a,x),(a,y)) = 2$, it follows that $[(a,x),(a,z),(a,y)]$ is an (a,x)-(a,y) geodesic in $K_n[G]$. This implies that $[x,z,y]$ is an x-y geodesic in G. Thus, $z \in I_G[x,y] = I_G[S]$. This shows that S is geodetic set in G; hence, $g(G) \leq 2$ by definition. Further, since G is not the trivial graph, $g(G) \geq 2$. Therefore, $g(G) = 2$.

Now, let $u, v \in V(G)$ with $d_G(u,v) > 1$. Since $u, v \in I_G[x,y]$, it can easily be verified that $d_G(u,v) = 2$. Consequently, $diam(G) = 2$.

For the converse, suppose first that $G = \overline{K}_2$ and let $V(G) = \{w, z\}$. Pick $a \in V(K_n)$ and let $T = \{(a,w), (a,z)\}$. Since $wz \notin E(G)$, it follows that $d_{K_n[G]}((a,w),(a,z)) = 2$. Moreover, if $b \in V(K_n) \setminus \{a\}$, then

$$(b,w), (b,z) \in I_{K_n[G]}[(a,w),(a,z)].$$

This implies that $V(K_n[G]) = I_{K_n[G]}[T]$, that is, T is a geodetic set of $K_n[G]$. Therefore, $g(K_n[G]) = 2$.

Finally, suppose that G is connected and

$$g(G) = diam(G) = 2.$$

Let $U = \{u, v\}$ be a geodetic basis of G. Choose $a \in V(K_n)$ and consider $U_a = \{(a,u),(a,v)\}$. Since $G \neq K_2$, $uv \notin E(G)$. Now, because $diam(G) = 2$, it follows that $d_G(u,v) = 2$. Hence, if $x \in V(G) \setminus U$, then $[u,x,v]$ is a u-v geodesic since $x \in I_G[u,v]$. It follows that $[(a,u),(a,x),(a,v)]$ is an (a,u)-(a,v) geodesic, showing that $\{a\} \times V(G) \subseteq I_{K_n[G]}[U_a]$. Clearly, $[V(K_n) \setminus \{a\}] \times V(G) \subseteq I_{K_n[G]}[U_a]$. Thus, $V(K_n[G]) = I_{K_n[G]}[U_a]$, that is, U_a is a geodetic set of $K_n[G]$. Therefore, $g(K_n[G]) = 2$. □

Our next goal is to characterize those non-complete graphs G such that $g(K_n[G]) = 3$. To do this, we first define a concept in a connected graph which we find very useful to our ends.

Definition 2.3 Let H be a connected graph and $S \subseteq V(H)$. The *2-path closure* of S is

$$P_2[S] = S \cup \{x \in V(H) : x \in I_H[u,v] \text{ for some } u, v \in S \text{ with } d_H(u,v) = 2\}.$$

A subset $S \subseteq V(H)$ is *2-path closure absorbing* in H if $P_2[S] = V(H)$.

Example 2.4 Consider the graph H in Figure 1. If $S = \{1,3,5,8\}$, then $P_2[S] = S \cup \{2,4,6,7\} = V(H)$. Thus, S is a 2-path closure absorbing set in H.

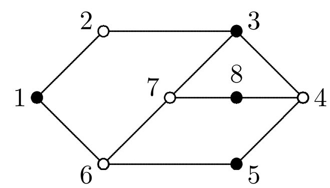

Figure 1: A graph H with a 2-path closure absorbing set

Theorem 2.5 *Let G be a non-complete graph and K_n the complete graph of order $n \geq 2$ such that $g(K_n[G]) \neq 2$. Then $g(K_n[G]) = 3$ if and only if one of the following holds:*

(*i*) $G = \overline{K}_3$;

(*ii*) $G = K_1 \cup H$, *where H is connected and* $g(H) = diam(H) = 2$;

(*iii*) *G is connected and there exists a subset $T \subseteq V(G)$ such that $|T| = 3$ and T is 2-path closure absorbing in G.*

Proof: Suppose $g(K_n[G]) = 3$. Let $S = \{(a,x),(b,y),(c,z)\}$ be a geodetic basis of $K_n[G]$. If a, b and c are all distinct, then the induced graph $\langle S \rangle$ of S is complete. This, however, is not possible because $K_n[G] \neq K_3$. Assume that $a = b$ and suppose that $c \neq a$. Then $(a,x)(c,z), (a,y)(c,z) \in$

$E(K_n[G])$. Again, since $K_n[G] \neq K_3$, $xy \notin E(G)$. It follows that $d_{K_n[G]}((a,x),(a,y)) = 2$. This implies that if $w \in V(G)$ with $xw \in E(G)$, then $(a,w) \in I_{K_n[G]}[(a,x),(a,y)]$; hence, $[(a,x),(a,w),(a,y)]$ is an (a,x)-(a,y) geodesic. Consequently, $[x,w,y]$ is an x-y geodesic. In other words, $d_G(x,y) = 2$. Suppose that $[V(G) \setminus \{x,y\}] \subseteq I_G[x,y]$. Then $(a,v) \in I_{K_n[G]}[(a,x),(a,y)]$ for all $v \in V(G)$. This means that $[\{a\} \times V(G) \subseteq I_{K_n[G]}[(a,x),(a,y)]$. Also, since

$$(d,v) \in I_{K_n[G]}[(a,x),(a,y)]$$

for all $d \in V(K_n) \setminus \{a\}$ and all $v \in V(G)$, it follows that $[V(K_n) \setminus \{a\} \times V(G)] \subseteq I_{K_n[G]}[(a,x),(a,y)]$. Thus, $S \setminus \{(c,z)\}$ is a geodetic set in $K_n[G]$, contrary to our assumption of S. Therefore, there exists $u \in V(G) \setminus \{x,y\}$ such that $u \notin I_G[x,y]$. This implies that $(a,u) \notin I_{K_n[G]}[(a,x),(a,y)]$. Since $(a,u) \notin I_{K_n[G]}[(a,x),(c,z)] \cup I_{K_n[G]}[(a,y),(c,z)]$, it follows that $(a,u) \notin I_{K_n[G]}[S]$. Therefore, S is not a geodetic set in $K_n[G]$, contrary to our assumption. Accordingly, $c = a$.

Now, since $a = b = c$, x, y, and z are distinct vertices of G. Let $T = \{x,y,z\}$. Suppose first that G is disconnected. Then G can only have at most three components. If G has exactly three components, then each of these components is the trivial graph. For, if one component is not trivial, then there exists $w \in V(G) \setminus \{x,y,z\}$ such that $(a,w) \notin I_{K_n[G]}[S]$, contrary to our assumption. Thus, $G = \overline{K}_3$. Suppose now that G contains two components H' and H. Suppose further that both components are non-trivial. Then again, there exists $w \in V(G) \setminus \{x,y,z\}$ such that $(a,w) \notin I_{K_n[G]}[S]$, contrary to our assumption. Consequently, one of these components, say H', is the trivial graph K_1. If $H = K_2$, then we are done. So, suppose $H \neq K_2$. Note that since $g(K_n[G]) \neq 2$, $G \neq \overline{K}_2$. This implies that $H \neq K_1$; hence, $g(H) \geq 2$. Without loss of generality, suppose that $V(H') = \{x\}$. Then $y, z \in V(H)$. Let $v \in V(H) \setminus \{y, z\}$. Clearly, $(a, v) \in I_{K_n[G]}[(a, y), ((a, z)]$, that is, $[(a, y), (a, v), (a, z)]$ is an (a, y)-(a, z) geodesic. Thus, $[y, v, z]$ is a y-z geodesic in H, showing that $\{y, z\}$ is a (minimum) geodesic set in H. Hence, $g(H) = 2$. Further, it is routine to show that $diam(H) = 2$. Therefore, conditions (i) and (ii) hold.

Next, suppose that G is connected and let $u \in V(G) \setminus T$. Then $(a, u) \in I_{K_n[G]}[S]$. This means that $(a, u) \in I_{K_n[G]}[(a, x), (a, y)] \cup I_{K_n[G]}[(a, x), (a, z)] \cup I_{K_n[G]}[(a, z), (a, y)]$. Assume that $(a, u) \in I_{K_n[G]}[(a, x), (a, y)]$. Since

$$d_{K_n[G]}((a, x), (a, y)) = 2,$$

it follows that $[(a, x), (a, u), (a, y)]$ is a geodesic; hence, $[x, u, y]$ is an x-y geodesic. This implies that $v \in P_2[T]$. Therefore, $V(G) = P_2[T]$, that is, T is a 2-path closure absorbing set in G.

For the converse, suppose first that (i) holds. Let $V(G) = \{u, v, w\}$ and put $S_1 = \{(a_0, u), (a_0, v), (a_0, w)\}$, where $a_0 \in V(K_n)$. Then $(a, x) \in I_{K_n[G]}[(a_0, u), (a_0, v)]$ for all $(a, x) \in V(K_n[G]) \setminus S_1$. Thus, $V(K_n[G]) = I_{K_n[G]}[S_1]$, that is, S_1 is a geodetic set in $K_n[G]$. Since $K_n[G] \neq K_1$ and $g(K_n[G]) \neq 2$, it follows that $g(K_n[G]) = |S_1| = 3$.

Next, suppose that (ii) holds. Let $V(K_1) = \{x_1\}$ and $T = \{x_2, x_3\}$ a geodetic basis of H. Choose $b \in V(K_n)$ and put $S_2 = \{(b, x_1), (b, x_2), (b, x_3)\}$. Clearly, $(a, x) \in I_{K_n[G]}[(b, x_1), (b, x_2)]$ for all $a \in V(K_n) \setminus \{b\}$ and all $x \in V(G)$. Also, since $g(H) = diam(H) = 2$, it follows that $(b, y) \in I_{K_n[G]}[(b, x_1), (b, x_2)]$ for all $y \in V(H)$. This implies that $V(K_n[G]) = I_{K_n[G]}[S_2]$, that is, S_1 is a geodetic set in $K_n[G]$. By assumptions, $g(K_n[G]) = |S_2| = 3$.

Finally, suppose $V(G)$ has a 2-path closure absorbing subset T with $|T| = 3$, say $T = \{u, v, w\}$. Pick $a \in V(K_n)$ and consider $S_3 = \{(a, u), (a, v), (a, w)\}$. It is easy to show that $I_{K_n[G]}[S] = V(K_n[G])$, that is, S is a geodetic set in $K_n[G]$. Again, since $K_n[G] \neq K_1$ and $g(K_n[G]) \neq 2$, it follows that $g(K_n[G]) = |S_1| = 3$. This completes the proof of the theorem. □

The following results are direct consequences of Theorem 2.2.

Corollary 2.6 *For $n \geq 2$,*

$$g(K_n[P_m]) = \begin{cases} 2, & \text{if } m = 3 \\ 3, & \text{if } m = 4, 5 \\ 4, & \text{if } m \geq 6. \end{cases}$$

Corollary 2.7 *For $m \geq 4$,*

$$g(K_n[C_m]) = \begin{cases} 2, & \text{if } m = 4 \\ 3, & \text{if } m = 5 \\ 4, & \text{if } m \geq 6. \end{cases}$$

3 The Hull Number of the Composition $G[H]$

In [GC], the authors determined the hull number of the composition $G[H]$, where G is connected and H is a connected non-complete graph. In this section, we show that a similar result holds even if H is a disconnected graph.

In what follows, $N(x)$ denotes the set of neighbors of x in a connected graph G. We shall prove our result through the following simple lemmas.

Lemma 3.1 *Let G be a nontrivial connected graph, H a non-complete graph and $u, v \in V(H)$ with $uv \notin E(H)$. Let $x \in V(G)$ and let $A_x = \{(x, u), (x, v)\} \subseteq V(G[H])$. If $y \in N(x)$, then $(y, w) \in I_{G[H]}[A_x]$ for all $w \in V(H)$.*

Proof: Let $u, v \in V(H)$ with $uv \notin E(H)$, and $x \in V(G)$. Consider the set $A_x = \{(x,u),(x,v)\} \subseteq V(G[H])$. If $y \in N(x)$ and $w \in V(H)$, then $[(x,u),(y,w),(x,v)]$ is an (x,u)-(x,v) geodesic in $G[H]$. Thus, $(y,w) \in I_{G[H]}[(x,u),(x,v)]$. Therefore, $(y,w) \in I_{G[H]}[A_x]$ for all $w \in V(H)$. $\square$

Lemma 3.2 *Let G be a nontrivial connected graph, H a non-complete graph and $u, v \in V(H)$ with $uv \notin E(H)$. Let $x \in V(G)$ and $A_x = \{(x,u),(x,v)\} \subseteq V(G[H])$. If $N(x) \neq \emptyset$, then $(x,w) \in I^2_{G[H]}[A_x]$ for all $w \in V(H)$.*

Proof: Let $u, v \in V(H)$ with $uv \notin E(H)$, and $x \in V(G)$. Consider the set $A_x = \{(x,u),(x,v)\} \subseteq V(G[H])$. Obviously, $(x,u),(x,v) \in I^2_{G[H]}[A_x]$.

Let $y \in N(x)$. By Lemma 3.1, $(y,w) \in I_{G[H]}[A_x]$ for all $w \in V(H)$. Let $B = \{(y,u),(y,v)\}$. Then $B \subseteq I_{G[H]}[A_x]$. It follows that $I_{G[H]}[B] \subseteq I_{G[H]}[I_{G[H]}[A_x]] = I^2_{G[H]}[A_x]$. Since $x \in N(y)$, $(x,w) \in I_{G[H]}[B]$ for all $w \in V(H)$ by Lemma 3.1. Therefore, $(x,w) \in I^2_{G[H]}[A_x]$ for all $w \in V(H)$. $\square$

Lemma 3.3 *Let G be a nontrivial connected graph, H a non-complete graph and $u, v \in V(H)$ with $uv \notin E(H)$. Further, let $x \in V(G)$ and set $A_x = \{(x,u),(x,v)\}$. For each $r \geq 1$, if $y \in V(G)$ and $d_G(x,y) = r$, then $(y,w) \in I^r_{G[H]}[A_x]$ for all $w \in V(H)$.*

Proof: Let $u, v \in V(H)$ with $uv \notin E(H)$, and

$$A_x = \{(x,u),(x,v)\} \subseteq V(G[H])$$

for $x \in V(G)$. By Lemma 3.1, the assertion holds for $r = 1$.

Suppose the assertion holds for $r = k$ $(k \geq 2)$, that is, if $y \in V(G)$ and $d_G(x,y) = k$, then $(y,w) \in I^k_{G[H]}[A_x]$ for all $w \in V(H)$. Suppose $y \in V(G)$ and $d_G(x,y) = k+1$. Consider an x-y geodesic $P_{k+1} = [x, x_1, x_2, \ldots, x_k, y]$. Since

$d_G(x, x_k) = k$, $(x_k, w) \in I^k_{G[H]}[A_x]$ for all $w \in V(H)$ by the inductive hypothesis. Let $B_k = \{(x_k, u), (x_k, v)\}$. Then $B_k \subseteq I^k_{G[H]}[A_x]$; hence, $I_{G[H]}[B_k] \subseteq I^{k+1}_{G[H]}[A_x]$. Since $y \in N(x_k)$, we have $(y, w) \in I_{G[H]}[B_k]$ for all $w \in V(H)$ by Lemma 3.1. Thus, $(y, w) \in I^{k+1}_{G[H]}[A_x]$ for all $w \in V(H)$. This completes the proof of the lemma. □

Theorem 3.4 *Let G be a connected nontrivial graph and H a non-complete graph. Then $h(G[H]) = 2$.*

Proof: Let G and H be connected graphs, where H is non-complete. If $G = K_1$, then $G[H] \cong H$. This implies that, $h(G[H]) = h(H)$.

Suppose $G \neq K_1$. Since H is non-complete, there exist $u, v \in V(H)$ such that $uv \notin E(H)$. Choose $x \in V(G)$ and let $A_x = \{(x, u), (x, v)\}$. Since $N(x) \neq \emptyset$, $(x, w) \in I^2_{G[H]}[A_x] \subseteq [A_x]$ for all $w \in V(H)$ by Lemma 3.2. Now, let $y \in V(G)\backslash\{x\}$ and let $d_G(x, y) = r$ $(r \geq 1)$. By Lemma 3.3, $(y, w) \in I^r_{G[H]}[A_x] \subseteq [A_x]$ for all $w \in V(H)$. Since $[A_x] \subseteq V(G[H])$, it follows that $[A_x] = V(G[H])$. Therefore, $h(G[H]) = 2$. □

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