MINDANAWAN Journal of Mathematics
Volume 1 ISSN 2094-7380 October 2011

TABLE OF CONTENTS

Domination in the K_r - gluing of Complete Graphs and Join of Graphs

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Abstract: In this paper, we characterized the dominating sets, total dominating sets and secure total dominating sets in the K_r - gluing of complete graphs. The dominating sets and total dominating sets in the join of two graphs were also investigated. As direct consequences, the domination, total domination and secure total domination numbers of these graphs were obtained

1 Introduction

Chartrand and Lesniak [3] and Haynes, Hedetnieme and Slater [5] believed and pointed out that domination in graphs actually started in the 1850's. This concept was formalized in 1958 with Berge [2] and Ore [6] in 1962. E. J. Cockayne et. al. [4] introduced the concept of total domination in graphs, which is one of the variants of domination. Recently, another variant of dominating sets was introduced in 2007 by Benecke et. al. [1] which is the so-called secure total dominating set.

In this paper, we characterized the dominating sets, total dominating sets and secure total dominating sets in the K_r - gluing of complete graphs. The dominating sets and total dominating sets in the join of two graphs were also investigated. As quick consequences, we determined the domination, total domination and secure total domination numbers of these graphs. To consider the results obtained, we need the following definitions.

Let $G = (V(G), E(G))$ be a connected graph and $v \in V(G)$. The neighborhood of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $X \subseteq V(G)$, then the *open neighborhood* of X is the set $N_G(X) = N(X) = \cup_{v \in X} N_G(v)$. The *closed neighborhood* of X is $N_G[X] = N[X] = X \cup N(X)$.

A subset X of $V(G)$ is a *dominating set* of G if for every $v \in (V(G)\backslash X)$, there exists $x \in X$ such that $xv \in E(G)$, i.e., $N[X] = V(G)$. It is a *total dominating set* if $N(X) = V(G)$. A total dominating set X is a *secure total dominating set* if for every $u \in V(G)\backslash X$, there exists $v \in X$ such that $uv \in E(G)$ and $[X\backslash\{v\}] \cup \{u\}$ is a total dominating set. In this case, we say that v *X-defends* u or u is *X-defended by* v. The *domination number* $\gamma(G)$ (*total domination number* $\gamma_t(G)$ and *secure total domination number* $\gamma_{st}(G)$) of G is the smallest cardinality of a dominating (resp., total dominating and secure total dominating) set of G. Clearly, we state the following remarks.

Remark 1.1 *For any graph G.*

1. $\gamma(G) \geq 1$.
2. $\gamma_t(G) \geq 2$.
3. $\gamma(G) \leq \gamma_t(G) \leq \gamma_{st}(G)$.

Let K_{p_1}, K_{p_2},…,K_{p_n} be some complete graphs, each containing a complete graph K_r for some integer r, $r < p_i$ for all i. The graph G obtained from the union of these n complete graphs by identifying the $K_{r'}s$ (one from each complete graph) in an arbitrary way is called the K_r - *gluing* of the complete graphs K_{p_1}, K_{p_2},...,K_{p_n}.

The *join* $G+H$ of two graphs G and H is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set

$$E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

2 Domination in the K_r - *gluing* of Complete Graphs

Theorem 2.1 *Let G be the K_r - gluing of the complete graphs K_{p_1}, $K_{p_2}, \dots, K_{p_n}$. Then $C \subseteq V(G)$ is a dominating set in G if and only if either*

(i) $C \cap V(K_r) \neq \varnothing$, or

(ii) $C \cap [V(K_{p_i}) \setminus V(K_r)] \neq \varnothing$ for every i.

Proof. Let $C \subseteq V(G)$ be a dominating set of G. If $C \cap V(K_r) \neq \varnothing$, then we are done. So, suppose $C \cap V(K_r) = \varnothing$ and $C \cap V(K_{p_i}) = \varnothing$ for some i. Let $x \in C$. Then for every $z \in V(K_{p_i}) \setminus V(K_r)$, $xz \notin E(G)$. This contradicts to our assumption of the set C. It follows that, $C \cap V(K_{p_i}) \neq \varnothing$ for every i.

For the converse, we first assume that (i) holds and let $x \in C \cap V(K_r)$. Then for every $y \in V(G)$, $xy \in E(G)$. Hence, C is a dominating set of G. Next, we assume (ii) holds and let $x_i \in C \cap V(K_{p_i}) \setminus V(K_r)$ for every i. Then, clearly $x_iy_i \in E(G)$ for every $y_i \in V(K_{p_i})$. This means that C is a dominating set of G. ■

Corollary 2.2 *Let G be the K_r - gluing of the complete graphs $K_{p_1}, K_{p_2}, \dots, K_{p_n}$. Then $\gamma(G) = 1$.*

Proof. Let $x \in K_r$. Then by Theorem 2.1, $C = \{x\}$ is a dominating set of G. It follows that $\gamma(G) \leq |C| = 1$. Therefore, by (1) of Remark 1.1, $\gamma(G) = 1$. ■

3 Total Domination in the K_r - gluing of Complete Graphs

Theorem 3.1 *Let G be the K_r - gluing of the complete graphs $K_{p_1}, K_{p_2}, \dots, K_{p_n}$. Then $C \subseteq V(G)$ is a total dominating set in G if and only if $|C| \geq 2$ and either*

(i) $C \cap V(K_r) \neq \varnothing$, or

(ii) $|C \cap V(K_{p_i})| \geq 2$ for every i.

Proof. Let C be a total dominating set of G. If $C \cap V(K_r) \neq \varnothing$ then we are done. So, suppose $C \cap V(K_r) = \varnothing$. If there exists i such that $C \cap V(K_{p_i}) = \varnothing$, then for every $z \in V(K_{p_i}) \setminus V(K_r)$, $xz \notin E(G)$ for every $x \in C$. This is not possible since C is a dominating set of G. It follows that $C \cap V(K_{p_i}) \neq \varnothing$ for every i. Suppose $|C \cap V(K_{p_i})| = 1$ for some i and let $y \in C \cap V(K_{p_i})$. Then for every $t \in V(G) \setminus V(K_{p_i})$, $yt \notin E(G)$. This is also impossible since C is a total dominating set of G. Therefore, $|C \cap V(K_{p_i})| \geq 2$ for every i.

For the converse, we first assume that (i) holds and let $v \in C \cap V(K_r)$. Since $|C| \geq 2$, we assume $w \in C$ with $v \neq w$. Then $vw \in E(G)$ and $uv \in E(G)$ for every $u \in V(G)$. It follows that C is a total dominating set of G. Next, if we assume that (ii) holds, then clearly, C is a total dominating set of G. ■

Corollary 3.2 *Let G be the K_r - gluing of the complete graphs $K_{p_1}, K_{p_2}, \ldots, K_{p_n}$. Then $\gamma_t(G) = 2$.*

Proof. Let $C = \{x, y\}$ where $x \in V(K_r)$ and $y \in V(G)$ with $x \neq y$. Then by Theorem 3.1, C is a total dominating set of G. Thus, $\gamma_t(G) \leq |C| = 2$. Therefore, by (2) of Remark 1.1, $\gamma_t(G) = 2$. ∎

4 Secure Total Domination in the K_r - gluing of Complete Graphs

Theorem 4.1 *Let G be the K_r - gluing of the complete graphs $K_{p_1}, K_{p_2}, \ldots, K_{p_n}$. Then $C \subseteq V(G)$ is a secure total dominating set in G if and only if it satisfies at least one of the following:*

(i) $|C \cap V(K_r)| \geq 2$.

(ii) $C \cap V(K_r) = \varnothing$ *and* $|C \cap [V(K_{p_i}) \setminus V(K_r)]| \geq 2$ *for every* i.

(iii) $|C \cap V(K_r)| = 1$ *and* $|C \cap [V(K_{p_i}) \setminus V(K_r)]| \geq 1$ *for every* i.

Proof. Let C be a secure total dominating set in G. If $|C \cap V(K_r)| \geq 2$ then we are done. So, suppose $|C \cap V(K_r)| < 2$. Then either $C \cap V(K_r) = \varnothing$ or $|C \cap V(K_r)| = 1$. If the former holds then $C \cap [V(K_{p_i}) \setminus V(K_r)] \neq \varnothing$ for every i. Because, if there exists j such that $C \cap [V(K_{p_j}) \setminus V(K_r)] = \varnothing$, then for every $x \in C$, $xy \notin E(G)$ with $y \in V(K_{p_i}) \setminus V(K_r)$. This means that C is not a dominating set of G which is contrary to the fact that C is a secure total dominating set. If $|C \cap [V(K_{p_j}) \setminus V(K_r)]| = 1$ for some j, say $z \in C \cap [V(K_{p_j}) \setminus V(K_r)]$, then $xz \notin E(G)$ for every $x \in C$ where $x \neq z$. This is a contradiction that C is a total dominating set of G. It follows that $|C \cap [V(K_{p_i}) \setminus V(K_r)]| \geq 2$ for every i. Now, suppose $|C \cap V(K_r)| = 1$ and let $v \in C \cap V(K_r)$. If there exists j such that $C \cap [V(K_{p_j}) \setminus V(K_r)] = \varnothing$, then for every $w \in V(K_{p_j}) \setminus V(K_r)$, $(C \setminus \{v\}) \cup \{w\}$ is not a total dominating set of G. This is again a contradiction to the assumption of the set C. It follows that $|C \cap [V(K_{p_i}) \setminus V(K_r)]| \geq 1$ for every i.

The converse is straightforward. ∎

Corollary 4.2 *Let G be the K_r - gluing of the complete graphs $K_{p_1}, K_{p_2}, \ldots, K_{p_n}$. Then $\gamma_{st}(G) = 2$.*

Proof. Let $C = \{x, y\}$ such that $x, y \in V(K_r)$. Then by Theorem 4.1, C is a secure total dominating set of G. It follows that $\gamma_{st}(G) \leq |C| = 2$. By (2) and (3) of Remark 1.1, $\gamma_{st}(G) \geq 2$. Therefore, $\gamma_{st}(G) = 2$. ∎

5 Domination in the Join of Graphs

Theorem 5.1 *Let G and H be connected graphs. Then $C \subseteq V(G + H)$ is a dominating set in $G + H$ if and only if at least one of the following is true:*

(i) $C \cap V(G)$ *is a dominating set in* G.

(*ii*) *$C \cap V(H)$ is a dominating set in H.*

(*iii*) *$C \cap V(G) \neq \varnothing$ and $C \cap V(H) \neq \varnothing$.*

Proof. Let $C \subseteq V(G+H)$ be a dominating set of $G+H$. Clearly, $C \cap V(G) \neq \varnothing$ or $C \cap V(H) \neq \varnothing$. If $C \cap V(G)$ is a dominating set in G or $C \cap V(H)$ is a dominating set in H, then we are done. So, suppose $C \cap V(G)$ and $C \cap V(H)$ are not dominating sets of G and H, respectively. Suppose further that condition (*iii*) does not hold, that is $C \cap V(G) = \varnothing$ or $C \cap V(H) = \varnothing$. Then $C \subseteq V(H)$ or $C \subseteq V(G)$. Assume without loss of generality that $C \subseteq V(G)$. Then $C \cap V(G) = C$. By assumption, C is not a dominating set of G. This contradicts the assumption that C is a dominating set of $G+H$. Therefore, $C \cap V(G) \neq \varnothing$ and $C \cap V(H) \neq \varnothing$.

For the converse, suppose (*i*) holds. Let $v \in V(G+H)\backslash C$. If $v \in V(G)$, then there exists $x \in C \cap V(G) \subseteq C$ such that $xv \in E(G)$ (hence $xv \in E(G+H)$). If $v \in V(H)$, $yv \in E(G+H)$ for all $y \in C \cap V(G)$. Thus, C is a dominating set in $G+H$. Similarly, C is a dominating set in $G+H$ if (*ii*) holds. Suppose now that (*iii*) holds. Let $x \in C \cap V(G)$ and $y \in C \cap V(H)$. Then $V(H) \subseteq N_{G+H}(x)$ and $V(G) \subseteq N_{G+H}(y)$. Hence, $V(G+H) \subseteq N_{G+H}[C]$. Therefore, C is a dominating set in $G+H$. ■

Corollary 5.2 *Let G and H be connected graphs. Then*

$$\gamma(G+H) = \begin{cases} 1 & \text{if } \gamma(G)=1 \text{ or } \gamma(H)=1, \\ 2 & \text{if } \gamma(G)\neq 1 \text{ and } \gamma(H)\neq 1. \end{cases}$$

Proof. Suppose $\gamma(G) = 1$, say $C = \{x\}$ is a dominating set in G. By Theorem 5.1 (*i*), C is a dominating set of $G+H$. It follows that $\gamma(G+H) \leq |C| = 1$. By (1) of Remark 1.1, $\gamma(G+H) = 1$. Similarly, $\gamma(G+H) = 1$ if $\gamma(\mathrm{H}) = 1$.

Now, suppose $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. Let $x \in V(G)$ and $y \in V(H)$. Set $C = \{x, y\}$. Then by Theorem 5.1 (*iii*), C is a dominating set in $G+H$. Hence, $\gamma(G+H) \leq |C| = 2$. By assumption and Theorem 5.1 (*i*) and (*ii*), none of the singleton subsets of $V(G+H)$ is a dominating set in $G+H$. Therefore, $\gamma(G+H) = 2$. ■

6 Total Domination in the Join of Graphs

Theorem 6.1 *Let G and H be connected graphs. Then $C \subseteq V(G+H)$ is a total dominating set in $G+H$ if and only if it satisfies at least one of the following:*

(*i*) *$C \cap V(G)$ is a total dominating set in G.*

(*ii*) *$C \cap V(H)$ is a total dominating set in H.*

(*iii*) *$C \cap V(G) \neq \varnothing$ and $C \cap V(H) \neq \varnothing$.*

Proof. Let $C \subseteq V(G+H)$ be a total dominating set of $G+H$. Clearly, $C \cap V(G) \neq \varnothing$ or $C \cap V(H) \neq \varnothing$. If $C \cap V(G)$ is a total dominating set in G or $C \cap V(H)$ is a total dominating set in H, then we are done. So, suppose $C \cap V(G)$ and $C \cap V(H)$ are not total dominating sets of G and H, respectively. Suppose further that (*iii*) does not hold, that is $C \cap V(G) = \varnothing$ or $C \cap V(H) = \varnothing$. Then $C \subseteq V(H)$ or $C \subseteq V(G)$. Assume without loss of generality that

$C \subseteq V(G)$. Then $C \cap V(G) = C$. By assumption, C is not a total dominating set of G. This contradicts the assumption that C is a total dominating set of $G+H$. Therefore, $C \cap V(G) \neq \varnothing$ and $C \cap V(H) \neq \varnothing$.

For the converse, suppose (i) holds. Let $v \in V(G+H)$. If $v \in V(G)$, then there exists $x \in C \cap V(G) \subseteq C$ such that $xv \in E(G)$ (hence $xv \in E(G+H)$). If $v \in V(H)$, $yv \in E(G+H)$ for all $y \in C \cap V(G)$. Thus, C is a total dominating set in $G+H$. Similarly, C is a total dominating set in $G+H$ if (ii) holds. Suppose now that (iii) holds. Let $x \in C \cap V(G)$ and $y \in C \cap V(H)$. Then $V(H) \subseteq N_{G+H}(x)$ and $V(G) \subseteq N_{G+H}(y)$. Hence, $V(G+H) \subseteq N_{G+H}(C)$. Therefore, C is a total dominating set in $G+H$. ■

Corollary 6.2 *Let G and H be connected graphs. Then $C = \{x, y\}$, where $x \in V(G)$ and $y \in V(H)$, is a minimum total dominating set in $G+H$ and $\gamma_t(G+H) = 2$.*

Proof. Let $x \in V(G)$ and $y \in V(H)$. Set $C = \{x, y\}$. Then by Theorem 6.1 (iii), C is a total dominating set in $G+H$. Hence, $\gamma_t(G+H) \leq 2$. By (2) of Remark 1.1, C is a minimum total dominating set in $G+H$ and $\gamma_t(G+H) = 2$. ■

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