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Local Stability Analysis: Predator-Prey System with Constant Rate of Harvesting

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Abstract

The effect of constant-rate harvesting has been investigated by many authors and there are very rich and interesting behaviors in the dynamics of the predator-prey system. This paper studies the effect of harvesting on the predator-prey model of N.H.Mohammad. It is assumed in this study that the prey outside the refuge and predator are harvested at constant rates. The stability of the system, when prey inside the refuge is growing exponentially, is analyzed explicitly. The bifurcation point has been computed and numerical simulations are made in this paper.

1 Introduction

The Predator-prey model has been studied by many mathematicians, biologists, and ecologists. In 1920's, the first Predator-prey model has been proposed by Alfred Lotka and permanent oscillation of the population has been shown [5]. In 1926, the same model has been studied and developed by Italian mathematician Vito Volterra [18]. The history of the model has been developed through the variations and extensions proposed in the 1930s, 1950s, and 1970s [9]. In [7], a simple predator–prey interaction was studied where the predator population is subjected to harvesting. Gause predator-prey models have been analyzed by Martin and Ruan [12] where the prey is harvested at a constant rate. Kar [10] considered the predator-prey model with the predator harvested and suggested that studying the combined harvesting of predator and prey population models is ideal. Later, Toaha and Hassan [16] investigated and studied this suggestion. The prey response to predators have been the center of the study of Andrew Sih in 1987-1988 [13]. He, together with Petranka and Kats, later examined the effect of the cost of prey refuge use on the Lotka-Voltera type of predator-prey system [14]. Later in 1993, R. Serquiña modified A. Sih et. al's model [14] by incorporating the four fundamental demographic parameters-birth, natural death, immigration, and emigration. In 2018, N.H.R. Mohammad made a local stability analysis of Serquiña's model with refuge when birth and death are being considered. Models of predator-prey interactions with refuge have been formulated when considering functional responses such as Holling Type II and Ratio-Dependent Functional Response.

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This paper seeks to modify the N.H. Mohammad's predator-prey model by incorporating a constant rate of harvesting. Also, to investigate the persistence or coexistence condition of the predator and prey population. Moreover, the author only considers the case where the growth of prey inside the refuge is exponential. The center of the analysis is the persistence or coexistence of both predator and prey when harvested. Stability analysis is done by nondimensionalization and linearization techniques. Routh-Hurwitz Criterion is also used to simplify the solution of doing the stability analysis. Further investigation and verification is done using Python 3.7 as a numerical method of the study.

2 Preliminaries

Definition 2.1. [17] Let $\vec{x} \in \mathbb{R}^n$. Consider equations of the form

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) \tag{1}$$

in which the independent variable t does not occur explicitly. A vector equation of this form is called **autonomous**. A point $\vec{a} \in \mathbb{R}^n$ with $f(\vec{a}) = \vec{0}$ is an **equilibrium solution** of the equation (1).

Definition 2.2. [6] An equilibrium solution $\vec{x}(t) = \vec{x}^*$ of equation (1) is **asymptotically stable (locally asymptotically stable)** if every solution $\vec{x} = \vec{\psi}(t)$ of equation (1) which starts sufficiently close to $\vec{x}^*$ at time $t = 0$ not only remains close to $\vec{x}^*$ for all future time, but ultimately approaches $\vec{x}^*$ as t approaches infinity.

Theorem 2.3. [6] *Let $\vec{x^*}$ be an equilibrium solution of equation (1).*

i) If all the eigenvalues of $J\vec{f}$ are negative, then $\vec{x^*}$ is locally asymptotically stable.

ii) If at least one of the eigenvalues of of $J\vec{f}$ is positive, then $\vec{x^*}$ is unstable.

Theorem 2.4. [11] ***(The Routh-Hurwitz Criterion)*** *The polynomial equation*

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{2}$$

has only roots with negative real parts if and only if $a_1 > 0,\ \ a_3 > 0,\ \ a_1a_2 > a_3$.

Theorem 2.5. [15] *Consider the characteristic polynomial equation (2).*

*i) If the three roots are real and negative, the equilibrium is a **attracting node**.*

*ii) If the two roots are complex conjugate with negative real parts and the other root is real and negative, the equilibrium is a **spiral node (attracting)**.*

*iii) If the three roots are real and positive, the equilibrium is a **node (repellor)**.*

*iv) If the two roots are complex conjugate with positive real part and the other root is real and positive, the equilibrium is a **spiral repellor**.*

*v) If the two roots are real and negative and the other root is real and positive, the equilibrium is a **saddle point index 1**.*

*vi) If the two roots are real and positive and the other root is real and negative, the equilibrium is a **saddle point index 2**.*

*vii) If the two roots are complex conjugate with negative real parts and the other root is real and positive, the equilibrium is a **spiral saddle index 1**.*

*viii) If the two roots are complex conjugate with positive real parts and the other root is real and negative, the equilibrium is a **spiral saddle index 2**.*

3 Predator-Prey System with Constant Harvesting Rate

Consider the model with Holling Type II Functional Response of N.H. Mohammad given by

$$\begin{aligned} \frac{dR}{dT} &= -eR + bQ + gR \\ \frac{dQ}{dT} &= eR - bQ - \frac{acQP}{1+mQ} \\ \frac{dP}{dT} &= \frac{acnQP}{1+mQ} - rP, \end{aligned}$$

where R is the number of prey inside the refuge, Q is the number of prey outside the refuge, P is the density of predators, a is the attack rate, b is the overall rate of prey return to refuge, c is the capture success rate, ac is the predation rate, e is the emergence rate, g is the intrinsic growth of increase ($g > 0$), m be the constant handling time for each prey captured, n is the conversion rate of consumed into the predator reproduction rate, and r is the mortality rate of predator. Note that $(R, P, Q) \in \mathbb{R}^3_+$, $a, b, c, e, g, m, n, r \in \mathbb{R}_+$.

When prey outside refuge and predator are harvested with constant rates H_q and H_p, respectively, the model is formulated as follows:

$$\begin{cases} \frac{dR}{dT} = -eR + bQ + gR \\ \frac{dQ}{dT} = eR - bQ - \frac{acQP}{1+mQ} - H_q \\ \frac{dP}{dT} = \frac{acnQP}{1+mQ} - rP - H_p \end{cases} \tag{3}$$

where H_q and H_p are positive.

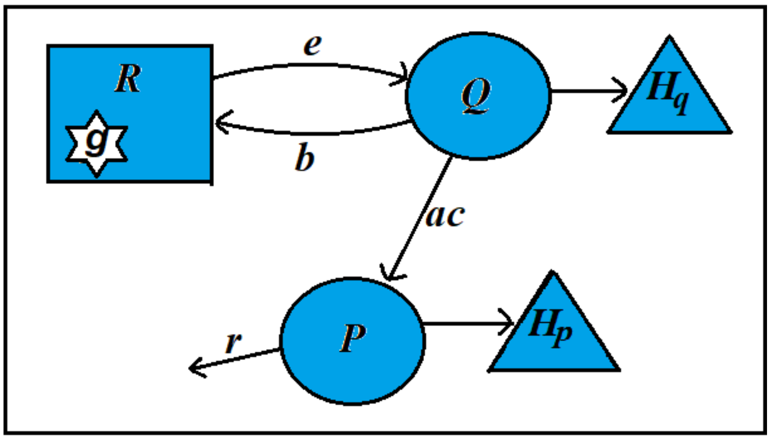

Figure 3.1: Compartmental Representation

Let $x = \frac{an}{b}R$, $y = \frac{an}{e}Q$, $z = \frac{ac}{e}P$, $t = eT$. Then, from model (3), the nondimensionalized system is

$$\frac{dx}{dt} = -\alpha x + y \tag{4a}$$

$$\frac{dy}{dt} = \sigma(x - y) - \frac{\beta yz}{\beta + y} - H_y \tag{4b}$$

$$\frac{dz}{dt} = \left(\frac{\delta y}{\beta + y} - 1\right)\phi z - H_z, \tag{4c}$$

where $e > g$ and $\alpha = \dfrac{e-g}{e}$, $\beta = \dfrac{an}{me}$, $\delta = \dfrac{acn}{mr}$, $\phi = \dfrac{r}{e}$, $\sigma = \dfrac{d}{e}$, $H_y = \dfrac{an}{e^2}H_q$ and $H_z = \dfrac{ac}{e^2}H_p$, are all positive constants.

Let (x^*, y^*, z^*) be the equilibrium point. For practical reasons, only the feasible equilibrium points are considered in this paper, that is, $x^* \geq 0, y^* \geq 0, z^* \geq 0$. Let

$$\left.\frac{dx}{dt}\right|_{x^*} = \left.\frac{dy}{dt}\right|_{y^*} = \left.\frac{dz}{dt}\right|_{z^*} = 0.$$

Since $H_z > 0$, it follows from equation (4c) that $(\delta - 1)y^* > \beta$ and so, $\delta > 1$. By equation(4a),

$$x^* = \frac{y^*}{\alpha}. \tag{5}$$

Substituting equation (5) to equation (4b), we obtain

$$z^* = \frac{[\sigma(1-\alpha)y^* - \alpha H_y](\beta + y^*)}{\alpha\beta y^*} \tag{6}$$

Substituting equation (6) to equation(4c) yields

$$a_1 y^{*2} + a_2 y^* + a_3 = 0,$$

where $a_1 = \sigma(1-\alpha)(\delta-1)$, $a_2 = \beta\sigma(1-\alpha) + \alpha H_y(\delta-1) + \dfrac{\alpha\beta}{\phi}H_z$, and $a_3 = \alpha\beta H_y$. Therefore, we get

$$y^* = \frac{a_2 \pm \sqrt{a_2^2 - 4a_1a_3}}{2a_1} \quad \text{and} \quad x^* = \frac{a_2 \pm \sqrt{a_2^2 - 4a_1a_3}}{2a_1\alpha}. \tag{7}$$

Lemma 3.1. *If $0 < \alpha < 1$, $\sigma(1-\alpha)y^* > \alpha H_y$, and $a_2^2 - 4a_1a_3 > 0$, then the equilibrium solution $E^*_{1,2} = (x^*, y^*, z^*)$ is feasible .*

Proof. Clearly, from equation (6), $z^* > 0$. Since $\delta > 1$ and $\sigma > 0$, it follows from equation (7) and equation (5) that $y^* > 0$ and $x^* > 0$ respectively, provided that $a_2^2 - 4a_1a_3 > 0$. Therefore, $E^*_{1,2}$ is feasible. □

System (4) has the Jacobian Matrix

$$J = \begin{bmatrix} -\alpha & 1 & 0 \\ \sigma & -r_1 & -r_2 \\ 0 & r_3 & r_4 \end{bmatrix}$$

where

$$r_1 = \sigma + \frac{\beta^2 z^*}{(\beta + y^*)^2}, r_2 = \frac{\beta y^*}{\beta + y^*}, r_3 = \frac{\beta\delta\phi z^*}{(\beta + y^*)^2} \text{ and } r_4 = \frac{\phi[(\delta-1)y^* - \beta]}{\beta + y^*}$$

are all nonnegative. Hence, the characteristic polynomial equation of the system is

$$\det(J - I\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \tag{8}$$

where $b_1 = r_1 - r_4 + \alpha$, $b_2 = (r_1 - r_4)\alpha + r_2r_3 - r_1r_4 - \sigma$, and $b_3 = (r_2r_3 - r_1r_4)\alpha + r_4\sigma$.

Theorem 3.2. *Let $(r_1 - r_4)\alpha + r_2r_3 - r_1r_4 > \sigma$. If $b_1b_2 > b_3$ such that $b_1 > 0$, $b_3 > 0$, then $E^*_{1,2}$ is locally asymptotically stable.*

Proof. From the Routh-Hurwitz Criterion, it follows that the roots of equation (8) are all negative. Therefore, $E^*_{1,2}$ is locally asymptotically stable by Theorem 2.3. □

Moreover, the roots of equation (8) are

$$\lambda_1 = A + B - \frac{b_1}{3} \tag{9}$$

$$\lambda_2 = -\frac{A+B}{2} - \frac{b_1}{3} + \frac{A-B}{2}\sqrt{-3} \tag{10}$$

$$\lambda_3 = -\frac{A+B}{2} - \frac{b_1}{3} - \frac{A-B}{2}\sqrt{-3} \tag{11}$$

where

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \quad B = \sqrt[3]{-\left(\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)} \tag{12}$$

such that

$$p = \frac{1}{3}(3b_2 - b_1^2,) \quad q = \frac{1}{27}(2b_1^3 - 9b_1b_2 + 27b_3),\ p, q \neq 0. \tag{13}$$

Throughout this paper, it is assumed that $p \geq -\sqrt[3]{\frac{q^2}{4}}$ to avoid the complex value of $\sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$.

Lemma 3.3. *Let $(r_1 - r_4)\alpha + r_2r_3 - r_1r_4 > \sigma$ such that $r_1 > r_4$ and $r_2r_3 > r_1r_4$. Then, any of the following holds;*

i) If $b_2 < \frac{b_1}{3}^2$ and $2b_1^3 + 27b_3 < 9b_1b_3$, then $p < 0$ and $q < 0$,

ii) If $b_2 > \frac{b_1}{3}^2$ and $2b_1^3 + 27b_3 > 9b_1b_3$, then $p > 0$ and $q > 0$,

iii) If $b_2 < \frac{b_1}{3}^2$ and $2b_1^3 + 27b_3 > 9b_1b_3$, then $p < 0$ and $q > 0$,

iv) if $b_2 > \frac{b_1}{3}^2$ and $2b_1^3 + 27b_3 < 9b_1b_3$, then $p > 0$ and $q < 0$.

Proof. Follows from equation (13) □

Lemma 3.4. *Let $(r_1 - r_4)\alpha + r_2r_3 - r_1r_4 > \sigma$ such that $r_1 > r_4$ and $r_2r_3 > r_1r_4$.*

i) Let $p < 0$ and $q < 0$ such that $p > -3\sqrt[3]{\frac{q^2}{4}}$. If $A + B > \frac{b_1}{3}$, then λ_1 is positive, otherwise λ_1 is negative. Also, λ_2 and λ_3 are complex conjugate with negative real part.

ii) Let $p = -3\sqrt[3]{\frac{q^2}{4}}$ and $q < 0$. If $A + B > \frac{b_1}{3}$, then λ_1 is positive, while λ_2 and λ_3 are real and negative, otherwise λ_1, λ_2, and λ_3 are all negative.

iii) Let $p > 0$ and $q \neq 0$. If $A > B - \frac{b_1}{3}$, then λ_1 is positive, otherwise λ_1 is negative. If $A > -B$, then λ_2 and λ_3 are complex conjugates with negative real part. If $-\frac{A+B}{2} > \frac{b_1}{3}$ such that $A < -B$, then λ_2 and λ_3 are complex conjugate with positive real part.

iv) Let $p < 0$ and $q > 0$ such that $p > -3\sqrt[3]{\frac{q^2}{4}}$. Then λ_1 is negative and real. If $-\frac{A+B}{2} > \frac{b_1}{3}$, then λ_2 and λ_3 are complex conjugate with positive real parts, otherwise the real parts are negative.

v) Let $p = -3\sqrt[3]{\frac{q^2}{4}}$ and $q > 0$. Then λ_1 is real and negative. If $-\frac{A+B}{2} > \frac{b_1}{3}$, then λ_2 and λ_3 are real and positive, otherwise λ_1, λ_2, and λ_3 are real and negative.

Proof. Follows from (9), (10) and (11). □

Theorem 3.5. *Let $(r_1 - r_4)\alpha + r_2r_3 - r_1r_4 > \sigma$ such that $r_1 > r_4$ and $r_2r_3 > r_1r_4$.*

i) Let $p < 0$ and $q < 0$ such that $p > -3\sqrt[3]{\frac{q^2}{4}}$.
If $A + B < \frac{b_1}{3}$, then $E^*_{1,2}$ is a spiral node.
If $A + B > \frac{b_1}{3}$, then $E^*_{1,2}$ is a spiral saddle index 1.

ii) Let $q < 0$ and $p = -3\sqrt[3]{\frac{q^2}{4}}$.
If $A + B < \frac{b_1}{3}$, then $E^*_{1,2}$ is an attracting node.
If $A + B > \frac{b_1}{3}$, then $E^*_{1,2}$ is a saddle point index 1.

iii) Let $p > 0$ and $q < 0$. Then,$E^*_{1,2}$ is either
a spiral repellor if $A > B - \frac{b_1}{3}$ and $-\frac{A+B}{2} > \frac{b_1}{3}$ such that $A < -B$;or
a spiral saddle index 1 if $A > B - \frac{b_1}{3}$ and $A > -B$; or
a spiral saddle index 2 if $A < B - \frac{b_1}{3}$ and $-\frac{A+B}{2} > \frac{b_1}{3}$ such that $A < -B$; or
a spiral node if $A > B - \frac{b_1}{3}$ and $A > -B$.

iv) Let $p < 0$ and $q > 0$ such that $p > -3\sqrt[3]{\frac{q^2}{4}}$. If $-\frac{A+B}{2} > \frac{b_1}{3}$, then $E^*_{1,2}$ is a spiral saddle index 2. If $-\frac{A+B}{2} < \frac{b_1}{3}$, then $E^*_{1,2}$ is a spiral node.

v) let $p = -3\sqrt[3]{\frac{q^2}{4}}$ and $q > 0$. If $-\frac{A+B}{2} > \frac{b_1}{3}$, then $E^*_{1,2}$ is a saddle point index 2. If $-\frac{A+B}{2} < \frac{b_1}{3}$, then $E^*_{1,2}$ is an attracting node.

Proof. Follows from Lemma 3.3, Lemma 3.4 and Theorem 2.5. □

Theorem 3.6. *If $r_1 > r_4$, $r_2r_3 > r_1r_4$ and $(r_1 - r_4)\alpha + r_2r_3 - r_1r_4 < \sigma$, then $E^*_{1,2}$ is unstable. In particular, $E^*_{1,2}$ is either a saddle point index 2 or a spiral saddle index 2.*

Proof. Not all roots of (8) are negative by Routh-Hurwitz Criterion. Thus, $E^*_{1,2}$ is unstable by Theorem 2.3.

Since $b_2 < 0$, it follows from (13) that λ_1 is real and negative. Since $E^*_{1,2}$ is unstable, it is either a saddle point index 2 by Theorem 2.5 (vi) or a spiral saddle index 2 by Theorem 2.5 (viii). □

Corollary 3.7. *Let $r_1 > r_4$ and $r_2r_3 > r_1r_4$. If $A + B < \frac{b_1}{3}$ and $q < 0$ or $-\frac{A+B}{2} < \frac{b_1}{3}$ and $q > 0$ such that $-3\sqrt[3]{\frac{q^2}{4}} \leq p < 0$, then the bifurcation occurs at $\sigma = (r_1 - r_4)\alpha + r_2r_3 - r_1r_4$.*

Proof. Follows from Theorem 3.2, Theorem 3.5, and Theorem 3.6. ☐

4 Numerical Simulations

4.1 Illustration 1

Consider the model (3) and assume that $a = 1$, $b = 1.2$, $c = 0.8$, $e = 0.9$, $g = 0.6$, $m = 0.8$, $n = 1.05$, $r = 0.3$, $Hq = 0.01$, and $Hp = 0.02$. Then, the equilibrium solution is $E^*_{1,2} = (1.789, 0.596, 3.727)$. From the characteristic equation (8), we get $b_1b_2 = 1.252 > b_3$. Hence, by Theorem 3.2, the equilibrium solution is locally asymptotically stable (See Figure 3.2). Also, $p = -3.82$, $q = 3.08$, $-\frac{A+B}{2} = 1.14$, and $\frac{b_1}{3} = 1.18$. Therefore, by Theorem 3.5 (iv), $E^*_{1,2}$ is a spiral node (See Figure 3.3).

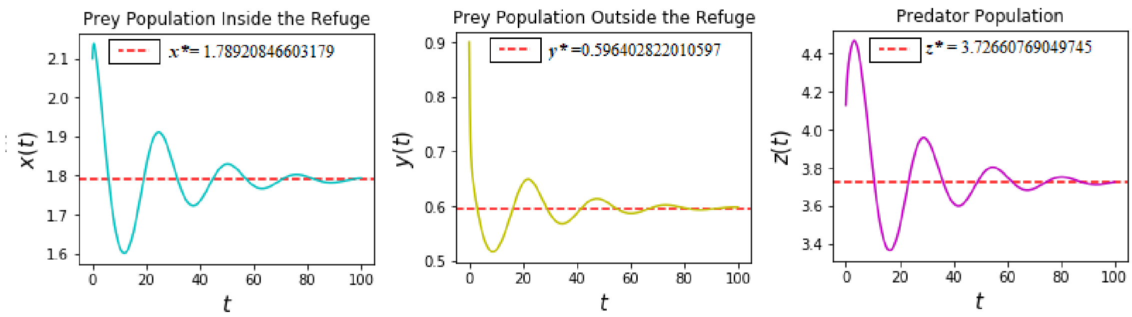

Figure 3.2: Asymptotic Behavior of the Predator-Prey System with initial values $x(0) = 2.1$, $y(0) = 0.9$, and $z(0) = 4.13$.

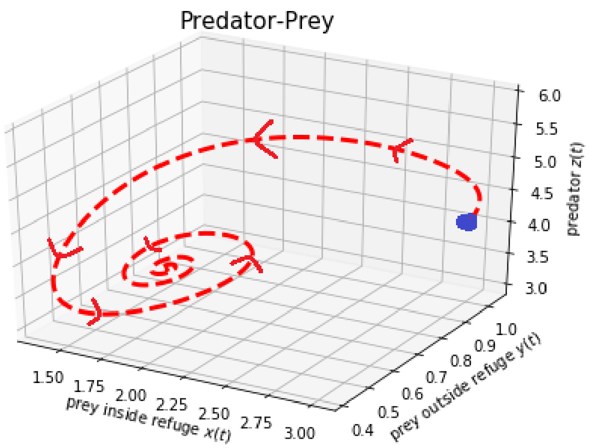

Figure 3.3: The Phase Space of the Predator-Prey System. The trajectory from the initial point $(2.1, 0.9, 4.13)$ approaches to the equilibrium point $E^*_{1,2}$.

4.2 Illustration 2

Let $a = 1$, $b = 1.2$, $c = 0.8$, $e = 0.9$, $g = 0.6$, $m = 0.8$, $n = 1.05$, $r = 0.3$, $Hq = 0.2$ and $Hp = 0.3$. Then, the equilibrium solution is $E^*_{1,2} = (2.41, 0.80, 3.64)$. Since $b_2 = -0.02$, it follows from Theorem 3.6 that $E^*_{1,2}$ is unstable. In Figure 3.4, the trajectory from the initial point $(2.5, 0.9, 3.7)$ goes away from the equilibrium point. Now, $p = -3.21$, $q = 2.46$, $-\dfrac{A+B}{2} = 1.05$, and $\dfrac{b_1}{3} = 1.03$. From Theorem 3.5 (iv), $E^*_{1,2}$ is a spiral saddle index 2.

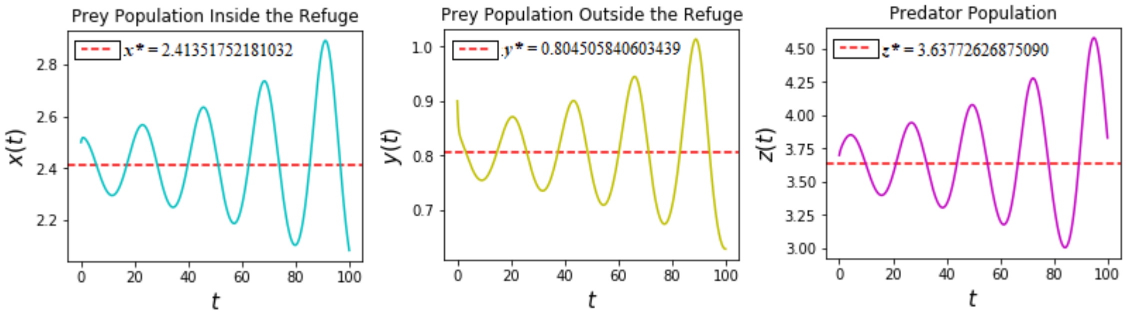

Figure 3.4: The Behavior of the Predator-Prey System with initial values $x(0) = 2.1$, $y(0) = 0.9$, and $z(0) = 4.13$. As t increases, the trajectory fluctuates and moves away from equilibrium points x^*, y^*, z^*.

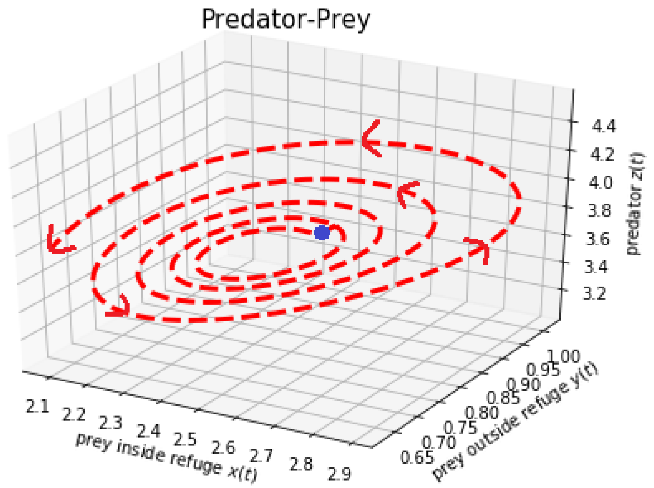

Figure 3.5: The Phase Space of the Predator-Prey System. The trajectory from the initial point $(2.5, 0.9, 3.7)$ goes away from the equilibrium point $E^*_{1,2}$.

5 Discussion and Conclusion

Harvest Management plays a significant role in real-world phenomena. For species that are managed for harvest, the manager must have an intimate knowledge of the species and its environment. In this paper N.H. Mohammad's predator-prey model is modified by considering the case that prey outside the refuge and predators are harvested at constant rates.

The persistence or coexistence condition of the predator and prey population has been investigated. Lemma 3.1 shows the feasibility of the equilibrium solution $E^*_{1,2}$ of system (3). Theorem 3.2 shows that $E^*_{1,2}$ can be locally asymptotically stable while in Theorem 3.6, $E^*_{1,2}$ can be unstable. Moreover, in Corollary 3.7, bifurcation point can occur at $\sigma = (r_1 - r_4)\alpha + r_2 r_3 - r_1 r_4$.

Finally, going through with the Numerical Simulations, it can be observed in Illustration 3.8 that when $H_q = 0.01$ and $H_p = 0.02$, the trajectory approaches to the equilibrium point $E^*_{1,2}$ (Figure 3.2 and Figure 3.3) and this implies that the predator-prey system is stable. However, with larger values $H_q = 0.2$ and $H_p = 0.3$, the trajectory is moving away from the equilibrium point (Figure 3.4 and Figure 3.5), which means that the Predator-Prey System becomes unstable. Hence, it is important to consider the value of harvesting management. If both predator and prey outside the refuge are harvested excessively, then both of these populations will surely face the danger of extinction.

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