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Some Properties of Holomorphic Functions having Convex Absolute Values and Applications

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Abstract

We study some properties concerning the convexity, plurisubharmonicity and other properties of certain special classes constructed from holomorphic functions. We prove that we have a key role between real and complex convexity in the theory of the representation of functions. On the other hand, let $g : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic nonconstant function. We prove that $|g|^2$ is convex on $\mathbb{C}^n$ if and only if g has a classical holomorphic representation. Several applications of this criterion are obtained in the theory consisting of convex and strictly plurisubharmonic functions, convex and strictly plurisubharmonic but not strictly convex functions and related topics.

1 Introduction

From Abidi [2], we can prove the following.

Lemma 1.1. *Let $u : \mathbb{C}^n \to [-\infty, +\infty[$ be a function, $n \geq 1$. Put $v(z, w) = u(w - \overline{z})$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. The following conditions are equivalent*

(a) *$u(\mathbb{C}^n) \subset \mathbb{R}$ and u is convex on $\mathbb{C}^n$;*

(b) *v is plurisubharmonic (psh) on $\mathbb{C}^n \times \mathbb{C}^n$.*

Observe that in the case where $u : \mathbb{C}^n \to \mathbb{R}$, and $k \in \mathbb{N}$, we have the equivalence between the following two technical conditions.

(c) u is convex and of class C^k on $\mathbb{C}^n$, and

(d) v is plurisubharmonic and of class C^k on $\mathbb{C}^n \times \mathbb{C}^n$.

Lemma 1.1 has many applications for the development of the theory of real and complex convexity and others complex analysis problems.

Several questions concerning the classes; convex and strictly plurisubharmonic functions, convex strictly plurisubharmonic but not strictly convex on all Euclidean not empty open balls,

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convex and not strictly psh on all open balls, convex strictly psh but not strictly convex, convex not strictly psh but not strictly convex at all points and many more related topics can be studied in [2].

Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions, $n \geq 1$ and $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$. Define $u(z,w) = |A_1 w - g_1(z)|^2 + |A_2 w - g_2(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. By [3], we prove that u is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if we have the holomorphic representation

$$\begin{cases} g_1(z) = A_1(< z/a > +b) + \overline{A_2}(< z/c > +d)^s \\ g_2(z) = A_2(< z/a > +b) - \overline{A_1}(< z/c > +d)^s \end{cases}$$

for all $z \in \mathbb{C}^n$ with $a, c \in \mathbb{C}^n$, $b, d \in \mathbb{C}$, $s \in \mathbb{N}$, or

$$\begin{cases} g_1(z) = A_1(< z/a_1 > +b_1) + \overline{A_2} e^{(<z/c_1>+d_1)} \\ g_2(z) = A_2(< z/a_1 > +b_1) - \overline{A_1} e^{(<z/c_1>+d_1)} \end{cases}$$

for all $z \in \mathbb{C}^n$, where $a_1, c_1 \in \mathbb{C}^n$, $b_1, d_1 \in \mathbb{C}$.

The following classes; convex and strictly psh functions, convex strictly psh and not strictly convex functions, convex strictly psh and not strictly convex in all not empty Euclidean open balls of $\mathbb{C}^n \times \mathbb{C}$ and many more, play a classical role in several problems of complex analysis and the theory of functions.

We consider the application of the holomorphic differential equation $k''(k + c) = \gamma(k')^2$ (where $k : \mathbb{C} \to \mathbb{C}$ is a holomorphic nonconstant function and $(\gamma, c) \in \mathbb{C}^2$) over a classical class of functions defined on $\mathbb{C}^n$, $n \geq 1$. The good condition $\gamma \in \{\frac{s-1}{s}, 1/s \in \mathbb{N}\backslash\{0\}\}$ is of great importance in all of this paper.

Using the above cited holomorphic differential equation, we prove that we have a classical relation between holomorphic partial differential equations and strictly plurisubharmonic functions on $\mathbb{C}^n$. We observe that we have a new proof of my result proved in [2] which is the following.

Let $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$ and $g : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. Using holomorphic differential equations, we prove that $|g + \alpha|$ and $|g + \beta|$ are convex functions on $\mathbb{C}^n$ if and only if g is an affine function on $\mathbb{C}^n$.

Moreover, we prove that in all bounded convex domains of $\mathbb{C}^n$, $n \geq 1$, this criterion is not true for several examples. At the end we prove several observations which are fundamental for proving technical questions between complex analysis and the theory of convex functions.

As usual, $\mathbb{N} := \{0, 1, 2, ...\}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all natural, real and complex numbers, respectively. Let U be a domain of $\mathbb{R}^d, d \geq 2$. We denote sh(U) the subharmonic functions on U and m_d the Lebesgue measure on $\mathbb{R}^d$. Let $f : U \to \mathbb{C}$ be a function. $|f|$ is the modulus of f, Re(f) and Im(f) are the real and imaginary parts of f respectively. For $N \geq 1$ and $h = (h_1, \ldots, h_N)$, where $h_1, \ldots, h_N : U \to \mathbb{C}$, $\| h \| = (|h_1|^2 + \ldots + |h_N|^2)^{\frac{1}{2}}$.

Let $g : D \to \mathbb{C}$ be a holomorphic function, D be a domain of $\mathbb{C}$. We denote by $g^{(0)} = g, g^{(1)} = g'$ which is the holomorphic derivative of g on D, $g^{(2)} = g'', g^{(3)} = g'''$. In general $g^{(m)} = \frac{\partial^m g}{\partial z^m}$ is the holomorphic derivative of g of order m, for all $m \in \mathbb{N}\backslash\{0\}$.
Let $z \in \mathbb{C}^n, z = (z_1, ..., z_n)$, For $n \geq 2$ and $j \in \{1, ..., n\}$, we write $(z = (z_j, Z_j) = (z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n))$ where $Z_j = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathbb{C}^{n-1}$. For $K : \mathbb{C}^n \to \mathbb{C}$, $K(z, .)$ is the function defined for $z_j \in \mathbb{C}$ by $K(., Z_j)(z_j) = K(z_j, Z_j) = K(z)$. Let $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$. We denote $< z/\xi >= z_1\overline{\xi_1} + \ldots + z_n\overline{\xi_n}$ and $B(\xi, r) = \{\zeta \in \mathbb{C}^n / \| \zeta - \xi \| < r\}$ for $r > 0$, where $\sqrt{< \xi/\xi >} = \| \xi \|$ is the Euclidean norm of ξ. We also consider the following notations:

$C^0(U) = \{\varphi : U \to \mathbb{C}/\varphi$ is continuous on $U\}$, $C^k(U) = \{\varphi : U \to \mathbb{C}/\varphi$ is of class C^k on $U\}$, and $C_c^\infty(U) = \{\varphi : U \to \mathbb{C}/\varphi \in C^\infty(U)$ and has a compact support on $U\}$, where $k \in \mathbb{N}\cup\{\infty\}$.

Let $\varphi : U \to \mathbb{C}$ be a function of class C^2, $\Delta(\varphi)$ is the Laplacian of φ. Let D be a domain of $\mathbb{C}^n, (n \geq 1)$. $psh(D)$ and $prh(D)$ are respectively the class of plurisubharmonic and pluriharmonic functions on D. For all $a \in \mathbb{C}, |a|$, $\text{Re}(a)$ and $\text{Im}(a)$ are the modulus, real and imaginary parts of a respectively. Also $D(a,r) = \{z \in \mathbb{C} \ / \ |z-a| < r\}$ and $\partial D(a,r) = \{z \in \mathbb{C} \ / \ |z-a| = r\}$, for $r > 0$. For a holomorphic polynomial p on $\mathbb{C}$, $\deg(p)$ is the degree of p.

For the study of properties and extension problems of holomorphic objects, we cite the references [1, 4, 5, 6, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 22, 23]. Moreover, several properties of holomorphic functions and their graphs are obtained in [8, 9]. The class of n-harmonic (or n-subharmonic) functions was introduced by Rudin in [21]. Good references for the study of convex functions in convex domains are [13, 16, 23].

2 Some fundamental analysis properties

In the sequel, using Abidi [2], we can now prove the following.

Lemma 2.1. *Let $f, g : \mathbb{C} \to \mathbb{C}$ be two holomorphic functions. Put*

$$u_1(z,w) = |wf(z) + g(z)|$$

$(z,w) \in \mathbb{C}^2$, $f \neq 0$. Suppose that u_1 is convex on $\mathbb{C}^2$. Then f is constant and g is an affine function on $\mathbb{C}$.

Proof. Let $u = u_1^2$. Then u is a function of class C^∞ and convex on $\mathbb{C}^2$. Hence $|\frac{\partial^2 u}{\partial z^2}(z,w)| \leq \frac{\partial^2 u}{\partial z \partial \overline{z}}(z,w)$, for all $(z,w) \in \mathbb{C}^2$. Note that
$u(z,w) = |w|^2|f(z)|^2 + |g(z)|^2 + g(z)\overline{w}\overline{f(z)} + \overline{g(z)}wf(z)$. Then

$$\begin{cases} \varphi(z,w) = |f''(z)\overline{f(z)}|w|^2 + g''(z)\overline{g(z)} + g''(z)\overline{w}\overline{f(z)} + \overline{g(z)}wf''(z)| \leq \\ |w|^2|f'(z)|^2 + |g'(z)|^2 + g'(z)\overline{w}\overline{f'(z)} + \overline{g'(z)}wf'(z) = \psi(z,w), \end{cases}$$

for all $(z,w) \in \mathbb{C}^2$. Observe that if $w = x_1 \in \mathbb{R} \backslash \{0\}$, we have

$$\lim_{x_1 \to +\infty} \frac{\varphi(z,x_1)}{x_1^2} \leq \lim_{x_1 \to +\infty} \frac{\psi(z,x_1)}{x_1^2}.$$

It follows that $|f''(z)\overline{f(z)}| \leq |f'(z)|^2$, for all $z \in \mathbb{C}$. This implies that $f''(z)f(z) = \gamma(f'(z))^2$, for each $z \in \mathbb{C}$, where $\gamma \in \mathbb{C}$. Then $|f|^2$ is convex on $\mathbb{C}$.

Now if $w_0 = 0$, then $u(., w_0)$ is convex on $\mathbb{C}$. It follows that $|g|^2$ is convex on $\mathbb{C}$. By Abidi [2], we have for all $z \in \mathbb{C}$,

$$f(z) = (az+b)^m, \text{ or } f(z) = e^{(a_1 z + b_1)}$$

and

$$g(z) = (cz+d)^s, \text{ or } g(z) = e^{(c_1 z + d_1)}$$

where $a, b, a_1, b_1, c, d, c_1, d_1 \in \mathbb{C}$, $m, s \in \mathbb{N}$.

Case 1. $f(z) = (az+b)^m$ and $g(z) = (cz+d)^s$, for all $z \in \mathbb{C}$.
Note that $u(z,w) = |w(az+b)^m + (cz+d)^s|^2$, $(z,w) \in \mathbb{C}^2$.

Assume that $m = 0$. It follows that $f = 1$. Now since $u(z,w) = |w + g(z)|^2$ and u is convex on $\mathbb{C}^2$, then g is an affine function on $\mathbb{C}$.

Suppose that $m = 1$. If $a = 0$, then $f = b^m$ on $\mathbb{C}$ and we conclude that g is an affine function on $\mathbb{C}$.

Suppose that $a \neq 0$. $u(z,w) = |w(az+b) + (cz+d)^s|^2$. Assume that $s = 1$. The condition $c = d = 0$, implies that $u(z,w) = |w(az+b)|^2$ and u is convex on $\mathbb{C}^2$. By a translation, we assume that $b = 0$. Thus $u(z,w) = |a|^2|wz|^2$, $(z,w) \in \mathbb{C}^2$. Take $w = z+1$, for $z \in \mathbb{C}$. These implies that $u(z, z+1) = |a|^2|z^2+z|^2 = K(z)$. But K is not convex on $\mathbb{C}$. Thus $c \neq 0$ or $d \neq 0$. Suppose that $c \neq 0$. Here, $u(z,w) = |w(az+b) + (cz+d)|^2$. Let $A \in \mathbb{C}\backslash\{0\}$, such that $[A(az+b)^2 + (cz+d)]$ has 2 zeros $z_0, z_1 \in \mathbb{C}$, $(z_0 \neq z_1)$.

Put $w_1(z) = A(az+b)$, for $z \in \mathbb{C}$. Then $u(z, w_1(z)) = |A(az+b)^2 + (cz+d)|^2 = K_1(z)$, $z \in \mathbb{C}$. Thus K_1 is convex on $\mathbb{C}$. But K_1 have two distinct zeros on $\mathbb{C}$. This is a contradiction.

Suppose that $m \geq 2$. Assume that $a \neq 0$. The case where $c = d = 0$ and $s \geq 1$ is impossible because $u(z,w) = |w(az+b)^m|^2$ which implies that u is not convex on $\mathbb{C}^2$. The case where $c \neq 0$ and $s \geq 1$ is also impossible. In fact we conclude that if $a \neq 0$, then u is not convex on $\mathbb{C}^2$. If $a = 0$ then f is constant on $\mathbb{C}$. Since $f \neq 0$, it follows that g is an affine function on $\mathbb{C}$.

The studies of the other cases are similar to case 1. □

Finally, in the sequel, we observe that we can prove the above lemma by a technical holomorphic differential equation on $\mathbb{C}$. Now in this section, we give an answer of the following question.

Question 2.2. Let $n \in \mathbb{N}$, $n \geq 3$. Does there exist an infinite number of holomorphic polynomials p, q on $\mathbb{C}$ such that $\deg(p) = \deg(q) = n$ and if we define $u_1(z,w) = |p(w-\overline{z})|^2$, $v_1(z,w) = |q(w-\overline{z})|^2$, $u_2(z,w) = |p'(w-\overline{z})|^2$, $v_2(z,w) = |q'(w-\overline{z})|^2$, $u = u_1 + v_1$, $v = u_2 + v_2$, $(z,w) \in \mathbb{C}^2$, then

$$\begin{cases} u_1 \text{ and } v_1 \text{ are functions not psh on } \mathbb{C}^2, \\ u_2 \text{ and } v_2 \text{ are functions not psh on } \mathbb{C}^2, \\ u \text{ is psh on } \mathbb{C}^2, \text{ and} \\ v \text{ is not psh on } \mathbb{C}^2? \end{cases}$$

Recall that we have by Abidi [2] the following result.

Theorem 2.3. *Let $u : B(a,R) \to \mathbb{R}$ be a continuous function, $a \in \mathbb{C}^n$, $R > 0$, $n \geq 1$. Define $G = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n / \parallel w - \overline{z} - a \parallel < R\}$ and $v(z,w) = u(w-\overline{z})$ for $(z,w) \in G$. (G is an open convex not bounded on $\mathbb{C}^n \times \mathbb{C}^n$). The following assertions are equivalent:*

(I) *u is convex on $B(a,R)$;*

(II) *v is psh on G.*

This theorem has technical applications in complex analysis and the theory of functions. Now several questions can be formulated from the above question. For example, at the end of this section we give an answer of the following.

Question 2.4. Let $n, k \in \mathbb{N}$, $n \geq 2$, $k \geq 1$. Is it true that there exists an infinite number of holomorphic polynomials p, q on $\mathbb{C}$ with $\deg(p) = \deg(q) = (n+k)$ and if we define $\varphi = u + v$, $\varphi_1 = u_1 + v_1, ..., \varphi_k = u_k + v_k$, where $u(z,w) = |p(w-\overline{z})|^2$, $v(z,w) = |q(w-\overline{z})|^2$, $u_1(z,w) = |p'(w-\overline{z})|^2$, $v_1(z,w) = |q'(w-\overline{z})|^2, ..., u_k(z,w) = |p^{(k)}(w-\overline{z})|^2$, $v_k(z,w) = |q^{(k)}(w-\overline{z})|^2$, for $(z,w) \in \mathbb{C}^2$. We have

$$\begin{cases} u, v, u_1, v_1, ..., u_k, v_k \text{ are functions not psh on } \mathbb{C}^2, \\ \varphi \text{ is psh on } \mathbb{C}^2, \\ \varphi_1 \text{ is not psh on } \mathbb{C}^2, \\ . \\ . \\ . \\ \varphi_k \text{ is not psh on } \mathbb{C}^2? \end{cases}$$

This is a technical investigation between the theory of holomorphic, convex and plurisubharmonic functions.

In this section, we prove the following result. Let p be a holomorphic polynomial on $\mathbb{C}$, $\deg(p) \geq 2$, $|p|$ not convex on $\mathbb{C}$. Then there exists an infinite number of holomorphic polynomials q on $\mathbb{C}$, $\deg(q) = 1$ and u is psh (or strictly psh) on $\mathbb{C}^2$, where $u(z,w) = |p(w-\overline{z})|^2 + |q(w-\overline{z})|^2$, for $(z,w) \in \mathbb{C}^2$. This result is not true for holomorphic functions in general.

On the other hand, let $\varphi = (|p_1|^2 + \ldots + |p_N|^2)$, where $p_1, \ldots, p_N$ are analytic polynomials on $\mathbb{C}$ and $N \geq 1$. Note that if $N = 1$, the assertion φ is convex on $\mathbb{C}$ implies that $4\frac{\partial^2\varphi}{\partial z \partial \overline{z}} = \Delta(\varphi)$ is convex on $\mathbb{C}$. We prove that this result is not true in general if $N \geq 2$.

Theorem 2.5. *Let $n \in \mathbb{N}$, $n \geq 3$. Then there exists an infinite number of analytic polynomials p, q on $\mathbb{C}$, deg(p) = deg(q) = n such that $|p'|^2$ and $|q'|^2$ are functions not convex on $\mathbb{C}$ and $u = (|p|^2 + |q|^2)$ is convex on $\mathbb{C}$, but $v = (|p'|^2 + |q'|^2)$ is not convex on $\mathbb{C}$.*

Proof. Assume that p_1 is an analytic polynomial on $\mathbb{C}$, $\deg(p_1) = n$ and $|p_1'|$ is not convex on $\mathbb{C}$. Define $p(w) = p_1(w) + Aw$, $q(w) = p_1(w) - Aw$ for all $w \in \mathbb{C}$, where $A \in \mathbb{R}_+ \backslash \{0\}$, A is to be constructed satisfying the following hypothesis: $(|p|^2 + |q|^2)$ is convex on $\mathbb{C}$ but $(|p'|^2 + |q'|^2)$ is not convex on $\mathbb{C}$. We have $p_1(w) = a_n w^n + ... + a_1 w + a_0$, for $w \in \mathbb{C}$, where $a_0, a_1, ..., a_n \in \mathbb{C}$, $a_n \neq 0$. Hence,

$$p_1'(w) = na_n w^{n-1} + ... + a_1, p_1''(w) = n(n-1)a_n w^{n-2} + ... + 2a_2,$$

and

$$(p_1'(w))^2 = n^2 a_n^2 w^{2n-2} + b_{2n-3} w^{2n-3} + ... + b_0,$$

with $b_0, ..., b_{2n-3} \in \mathbb{C}$. Also $p_1''(w)p_1(w) = n(n-1)a_n^2 w^{2n-2} + c_{2n-3} w^{2n-3} + ... + c_0$, where $c_0, ..., c_{2n-3} \in \mathbb{C}$. Then

$$\lim_{|w| \to +\infty} \frac{|p_1''(w)p_1(w)|}{|p_1'(w)|^2} = \frac{n-1}{n} < 1.$$

Therefore there exists $B > 0$ such that $|w| > B$ implies that $|p_1''(w)p_1(w)| <$ $|p_1'(w)|^2$.

Since now $\overline{D(0,B)}$ is compact on $\mathbb{C}$ and $|p_1''p_1|$ is a continuous function on $\overline{D(0,B)}$, there exists $M > 0$ such that $|p_1''(w)p_1(w)| < M$, for each $w \in \overline{D(0,B)}$.

Since, by Abidi [2], the cardinal of the set

$$\{\alpha \in \mathbb{C} \;/\; |p_1' + \alpha| \text{ is convex on } \mathbb{C}\}$$

is less than 1, we can choose $A > 0$ such that $A^2 \geq M$ and $|p_1' - A|$ and $|p_1' + A|$ are not convex functions on $\mathbb{C}$. Then we have $|p_1''(w)p_1(w)| < |p_1'(w)|^2 + A^2$, for each $w \in \mathbb{C}$. Thus

$$u(w) = (|p_1(w) + Aw|^2 + |p_1(w) - Aw|^2) = 2(|p_1(w)|^2 + A^2|w|^2)$$

for each $w \in \mathbb{C}$ and u is convex on $\mathbb{C}$. Now define $p(w) = p_1(w) + Aw$, $q(w) = p_1(w) - Aw$, for $w \in \mathbb{C}$. We have p and q are holomorphic polynomials on $\mathbb{C}$, $\deg(p) = \deg(q) = n \geq 3$, $|p'|$ and $|q'|$ are functions not convex on $\mathbb{C}$, but $u = (|p|^2 + |q|^2)$ is convex on $\mathbb{C}$. Note that $p'(w) = p_1'(w) + A$, $q'(w) = p_1'(w) - A$ and $|p_1'(w) + A|^2 + |p_1'(w) - A|^2 = 2(|p_1'(w)|^2 + A^2) = v(w)$, for $w \in \mathbb{C}$. Define $v_1(w) = |p_1'(w)|^2$, for $w \in \mathbb{C}$. Then v and v_1 are functions of class C^∞ on $\mathbb{C}$. Thus v_1 is convex on $\mathbb{C}$ if and only if v is convex on $\mathbb{C}$.

Since v_1 is not convex on $\mathbb{C}$. Consequently, v is not convex on $\mathbb{C}$. □

We have the following additional result.

Theorem 2.6. *Let $n \in \mathbb{N}$, $n \geq 3$. Then there exists an infinite number of holomorphic polynomials p, q on $\mathbb{C}$, $deg(p) = deg(q) = n$ and*

$$\begin{cases} |p'|^2 \text{ and } |q'|^2 \text{ are functions not convex on } \mathbb{C}; \\ (|p|^2 + |q|^2) \text{ is convex on } \mathbb{C}, \text{ and} \\ (|p'|^2 + |q'|^2) \text{ is convex on } \mathbb{C}. \end{cases}$$

Proof. Let p_1 be a holomorphic polynomial on $\mathbb{C}$, $\deg(p_1) = n$, $|p_1'|$ is convex on $\mathbb{C}$. We will construct p and q on the form $p(w) = p_1(w) + Aw$, $q(w) = p_1(w) - Aw$, for all $w \in \mathbb{C}$, where $A \in \mathbb{R}_+ \backslash \{0\}$, A is to be constructed satisfying the following hypotheses $u = (|p|^2 + |q|^2)$ and $v = (|p'|^2 + |q'|^2)$ are convex functions on $\mathbb{C}$, but $|p'|^2$ and $|q'|^2$ are not convex functions on $\mathbb{C}$.

We have $\lim\limits_{|w| \to +\infty} \dfrac{|p_1''(w) p_1(w)|}{|p_1'(w)|^2} = \dfrac{n-1}{n} < 1$. Then there exists $B > 0$ such that $|w| > B$ implies that $|p_1''(w)p_1(w)| < |p_1'(w)|^2$.

Now since $\overline{D}(0, B)$ is compact on $\mathbb{C}$ and the function $|p_1'' p_1|$ is continuous on $\overline{D}(0, B)$, there exists $\eta > 0$ such that $|p_1''(w)p_1(w)| < \eta$, for all $w \in \overline{D}(0, B)$. Recall that $\{\beta \in \mathbb{C} \ / \ |p_1' + \beta| \text{ is convex on } \mathbb{C}\}$ has a cardinal less than or equal to 1. Choose then $A > 0$, such that $A^2 \geq \eta$, $|p_1' - A|^2$ and $|p_1' + A|^2$ are functions not convex on $\mathbb{C}$. Put $p(w) = p_1(w) + Aw$ and $q(w) = p_1(w) - Aw$, for $w \in \mathbb{C}$. Note that p and q are holomorphic polynomials on $\mathbb{C}$.

Now we can verify that $|p'|^2$ and $|q'|^2$ are functions not convex on $\mathbb{C}$ but u and v are convex functions on $\mathbb{C}$. □

Corollary 2.7. *Let p be a holomorphic polynomial on $\mathbb{C}$, $deg(p) = n \geq 3$. Then there exists an infinite number of holomorphic polynomials p_1, q_1 on $\mathbb{C}$, $deg(p_1) = deg(q_1) = n$ such that*

$$\begin{cases} u = (|p_1|^2 + |q_1|^2) \text{ is convex on } \mathbb{C}, \\ \lim\limits_{|w| \to +\infty} \dfrac{|p(w)|}{|p_1(w)|} = \lim\limits_{|w| \to +\infty} \dfrac{|p(w)|}{|q_1(w)|} = 1. \end{cases}$$

Example 2.8. Let $p_1(w) = w^3 + w^2$, for $w \in \mathbb{C}$. There exists an infinite number of holomorphic polynomials p, q on $\mathbb{C}$, $\deg(p) = \deg(q) = 3$, $u = (|p|^2 + |q|^2)$ is convex on $\mathbb{C}$, but $v = (|p'|^2 + |q'|^2)$ is not convex on $\mathbb{C}$, $|p'|^2$ and $|q'|^2$ are functions not convex on $\mathbb{C}$,

$$\lim_{|w| \to +\infty} \frac{|p(w)|}{|p_1(w)|} = \lim_{|w| \to +\infty} \frac{|q(w)|}{|p_1(w)|} = 1.$$

Corollary 2.9. *Let p be an analytic polynomial on $\mathbb{C}$, $deg(p) = n \geq 2$. Then there exists an infinite number of $\alpha \in \mathbb{C}$ such that for all $\delta \in \mathbb{C}$, the function u_δ is convex on $\mathbb{C}$, $u_\delta(z) = |p(z)|^2 + |\alpha z + \delta|^2$, for $z \in \mathbb{C}$.*

Proof. Define $p_1(z) = p(z) + \alpha z$, $q_1(z) = p(z) - \alpha z$, (for $z \in \mathbb{C}$), where $\alpha \in \mathbb{C}$ is to be constructed satisfying the condition u is convex on $\mathbb{C}$, where
$u(z) = (|p_1(z)|^2 + |q_1(z)|^2) = 2(|p(z)|^2 + |\alpha z|^2)$ for $z \in \mathbb{C}$.

We have $\lim\limits_{|z| \to +\infty} \dfrac{|p''(z)p(z)|}{|p'(z)|^2} = \dfrac{n-1}{n} < 1$. Then there exists $B > 0$ such that for all $z \in \mathbb{C}$, $|z| > B$, we have $|p''(z)p(z)| < |p'(z)|^2$ and $(p'(z) \neq 0)$.

Now $\overline{D}(0, B)$ is compact of $\mathbb{C}$ and the function $|p''p|$ is continuous on $\overline{D}(0, B)$, then there exists $M > 0$ satisfying $|p''(z)p(z)| < M$, for all $z \in \overline{D}(0, B)$. Let $\alpha \in \mathbb{C}$, $|\alpha|^2 \geq M$. Then $|p''(z)p(z)| < |p'(z)|^2 + |\alpha|^2$. Thus v is convex on $\mathbb{C}$, where $v(z) = (|p(z)|^2 + |\alpha z|^2)$, for $z \in \mathbb{C}$. Therefore for all $\delta \in \mathbb{C}$, the function u_δ is convex on $\mathbb{C}$. □

Remark 2.10. For all analytic polynomials p on $\mathbb{C}$, where $\deg(p) = n \geq 2$ and $|p|$ is not convex on $\mathbb{C}$, there exists an infinite number of $\alpha \in \mathbb{C}$ such that u_δ is convex on $\mathbb{C}$, for all $\delta \in \mathbb{C}$, where $u_\delta(z) = (|p(z)|^2 + |\alpha z + \delta|^2)$, for $z \in \mathbb{C}$.

Lemma 2.11. *Let $g : \mathbb{C} \to \mathbb{C}$ be an analytic function. Assume that $g(w) = aw + b$ or $g(w) = \frac{1}{\gamma} e^{(\gamma w + \delta)} + \mu$ for $w \in \mathbb{C}$, where $a, b, \delta, \mu \in \mathbb{C}, a \neq 0$, $\gamma \in \mathbb{C}\backslash\{0\}, |\gamma| \neq 1$. Let q be an analytic polynomial on $\mathbb{C}$, deg$(q) = 1$. Define $u(z, w) = |e^{e^{g(w-\overline{z})}}|^2 + |q(w - \overline{z})|^2$, for $(z, w) \in \mathbb{C}^2$. Then u is not psh on $\mathbb{C}^2$.*

Proof. Define $v(w) = |e^{e^{g(w)}}|^2 + |q(w)|^2$, for $w \in \mathbb{C}$. We prove that v is not convex on $\mathbb{C}$. Assume that v is convex on $\mathbb{C}$. v is a function of class C^∞ on $\mathbb{C}$.

Case 1. Assume that $g(w) = aw + b$, for all $w \in \mathbb{C}$.

Then $|\frac{\partial^2 v}{\partial w^2}(w)| \leq \frac{\partial^2 v}{\partial w \partial \overline{w}}(w)$, for $w \in \mathbb{C}$. Thus, $|g''(w) + (g'(w))^2(1 + e^{g(w)})||e^{g(w)}||e^{e^{g(w)}}|^2 \leq |g'(w)|^2|e^{g(w)}|^2|e^{e^{g(w)}}|^2 + |q'(w)|^2$, for every $w \in \mathbb{C}$. $g'' = 0$ on $\mathbb{C}$. Let $(w_j)_{j\geq1} \subset \mathbb{C}$, $\lim\limits_{j\to+\infty} g(w_j) = +\infty$, $g(w_j) > 0$, for each $j \in \mathbb{N}$. Hence we have $|g'(w_j)|^2 e^{g(w_j)}(e^{e^{g(w_j)}})^2 \leq |q'(w_j)|^2$ and $g'(w_j) = a$, $q'(w_j) = \alpha \in \mathbb{C}$, for all $j \in \mathbb{N}$. Therefore the sequence of positive real numbers $(e^{g(w_j)}(e^{e^{g(w_j)}})^2)_{j\geq1}$ is bounded above. Since $\lim\limits_{j\to+\infty} g(w_j) = +\infty$, we have a contradiction.

Consequently, v is not a convex function on $\mathbb{C}$.

Case 2. Assume that $g(w) = \frac{1}{\gamma} e^{(\gamma w + \delta)} + \mu$, for each $w \in \mathbb{C}$.

Using the triangle inequality and the above proof, we prove that v_1 is not convex on $\mathbb{C}$, where $v_1(w) = |e^{e^{g(w)}}|^2 + |q(w)|^2$, for $w \in \mathbb{C}$. □

For holomorphic functions, we have the following.

Theorem 2.12. *There exists an analytic function $g : \mathbb{C} \to \mathbb{C}$ satisfying the hypothesis; $u = (|g|^2 + |q|^2)$ is not convex on $\mathbb{C}$, for any holomorphic polynomial q on $\mathbb{C}$ with degree less than 1.*

Proof. Consider $g(z) = e^{z^4}$, for $z \in \mathbb{C}$. Let q be an analytic polynomial on $\mathbb{C}$, with $\deg(q) \leq 1$. Then u is a function of class C^∞ on $\mathbb{C}$. Assume that u is convex on $\mathbb{C}$. Then $|\frac{\partial^2 u}{\partial z^2}(z)| \leq \frac{\partial^2 u}{\partial z \partial \overline{z}}(z)$, for each $z \in \mathbb{C}$. Thus

$$|12z^2 + 16z^6||e^{(z^4)}| \leq |16z^6||e^{z^4}| + |q'(z)|^2,$$

for each $z \in \mathbb{C}$. Hence, for all $z = x \in \mathbb{R}$, we have $\varphi(x) = 12x^2 e^{(x^4)} \leq |q'(x)|^2 = c$, where $c \in \mathbb{R}_+$. Thus, the function φ is bounded above on $\mathbb{R}$, which is a contradiction. Consequently, u is not convex on $\mathbb{C}$. □

Moreover, we have the technical investigation.

Theorem 2.13. *There exists $g : \mathbb{C} \to \mathbb{C}$, g is holomorphic and not affine on $\mathbb{C}$ such that for all holomorphic polynomials q on $\mathbb{C}$, we have u is not convex on $\mathbb{C}$, where $u(w) = (|e^{g(w)}|^2 + |q(w)|^2)$ for $w \in \mathbb{C}$.*

Proof. Let $g(w) = w^2$, for $w \in \mathbb{C}$. Then g is analytic and not affine on $\mathbb{C}$. Now let q be an analytic polynomial on $\mathbb{C}$. Define $u(w) = |e^{g(w)}|^2 + |q(w)|^2$, $w \in \mathbb{C}$. Then u is a function of class C^∞ on $\mathbb{C}$. Assume that u is convex on $\mathbb{C}$. We have then

$$|\frac{\partial^2 u}{\partial w^2}(w)| = |(4w^2 + 2)|e^{w^2}|^2 + q''(w)\overline{q}(w)| \leq |4w^2||e^{w^2}|^2 + |q'(w)|^2$$

for each $w \in \mathbb{C}$. For $w = x \in \mathbb{R}$, we have

$$(4x^2 + 2)e^{2x^2} - |q''(x)q(x)| \leq 4x^2 e^{2x^2} + |q'(x)|^2.$$

Therefore $2e^{2x^2} \leq |q''(x)q(x)| + |q'(x)|^2$, for each $x \in \mathbb{R}$. Thus $2 \leq \frac{|q''(x)q(x)|}{e^{2x^2}} + \frac{|q'(x)|^2}{e^{2x^2}}$, for every $x \in \mathbb{R}$. Since $\lim_{x\to+\infty} \dfrac{|q''(x)q(x)|}{e^{2x^2}} = 0$ and $\lim_{x\to+\infty} \dfrac{|q'(x)|^2}{e^{2x^2}} = 0$,

$$2 \leq \lim_{x\to+\infty} (\frac{|q''(x)q(x)|}{e^{2x^2}} + \frac{|q'(x)|^2}{e^{2x^2}}) = 0,$$

a contradiction. Consequently, u is not convex on $\mathbb{C}$. □

Theorem 2.14. *Let $n, k \in \mathbb{N}$, $n \geq 2, k \geq 1$. Then there exists an infinite number of analytic polynomials p, q on $\mathbb{C}$ with $deg(p) = deg(q) = (n + k)$ such that if we define $\varphi = u + v$, $\varphi_1 = u_1 + v_1, \ldots \varphi_k = u_k + v_k$, where $u(z, w) = |p(w - \overline{z})|^2$, $v(z, w) = |q(w - \overline{z})|^2$, $u_1(z, w) = |p'(w - \overline{z})|^2$, $v_1(z, w) = |q'(w - \overline{z})|^2, \ldots, u_k(z, w) = |p^{(k)}(w - \overline{z})|^2$, $v_k(z, w) = |q^{(k)}(w - \overline{z})|^2$, for $(z, w) \in \mathbb{C}^2$, then we have the system of assertions*

$$\begin{cases} u \text{ and } v \text{ are functions not psh on } \mathbb{C}^2, \\ u_1 \text{ and } v_1 \text{ are functions not psh on } \mathbb{C}^2, \\ \vdots \\ u_k \text{ and } v_k \text{ are functions not psh on } \mathbb{C}^2, \\ \varphi \text{ is psh on } \mathbb{C}^2, \\ \varphi_1 \text{ is not psh on } \mathbb{C}^2, \\ \vdots \\ \varphi_k \text{ is not psh on } \mathbb{C}^2. \end{cases}$$

Proof. Let p_1 be an analytic polynomial on $\mathbb{C}$, where $\deg(p_1) = n + k$ and $|p_1'|, \ldots, |p_1^{(k)}|$ are not convex functions on $\mathbb{C}$. Then, we have

$$\lim_{|w|\to+\infty} \frac{|p_1''(w)p_1(w)|}{|p_1'(w)|^2} = \frac{n + k - 1}{n + k} < 1.$$

Hence there exists $B > 0$ such that $|w| > B$ implies that $|p_1''(w)p_1(w)| < |p_1'(w)|^2$. Now B is fixed, $\overline{D}(0, B)$ is a compact subset of $\mathbb{C}$ and the function $|p_1''p_1|$ is continuous on $\overline{D}(0, B)$, then there exists $M > 0$ such that $|p_1''(w)p_1(w)| < M$, for all $w \in \overline{D}(0, B)$. Recall now that the cardinal of the set $\{\alpha \in \mathbb{C} \ / \ |p_1' + \alpha| \text{ is convex on } \mathbb{C}\}$ is less than or equal to 1 by Abidi [2]. Because p_1' is not an affine polynomial on $\mathbb{C}$, we can choose $A > 0$, $A^2 \geq M$ such that $|p_1' - A|$ and $|p_1' + A|$ are not convex functions on $\mathbb{C}$.

Hence we have $|p_1''(w)p_1(w)| < |p_1'(w)|^2 + A^2$, for every $w \in \mathbb{C}$. We now define $p(w) = p_1(w) + Aw$, $q(w) = p_1(w) - Aw$, for $w \in \mathbb{C}$. Then note that p and q are analytic polynomials on $\mathbb{C}$ and $\deg(p) = \deg(q) = (n+k)$. $\varphi(0, w) = (|p_1(w) + Aw|^2 + |p_1(w) - Aw|^2) = 2(|p_1(w)|^2 + |Aw|^2) = \psi(w)$, for $w \in \mathbb{C}$. Then the function ψ is convex on $\mathbb{C}$. By Abidi [2], φ is then psh on $\mathbb{C}^2$.

Here $|p'|^2 = |p_1' + A|^2$ is not convex on $\mathbb{C}$ and so $|p|^2$ is not convex on $\mathbb{C}$. Then u and u_1 are not psh functions on $\mathbb{C}^2$. Also since $|q'|^2 = |p_1' - A|^2$ is not convex on $\mathbb{C}$, then v and v_1 are not psh functions on $\mathbb{C}^2$. Now since $|p^{(k)}|^2 = |p_1^{(k)}|^2 = |q^{(k)}|^2$ on $\mathbb{C}$ and $|p_1^{(k)}|^2$ is not convex on $\mathbb{C}$, $|p^{(k)}|^2$ is not convex on $\mathbb{C}$. It follows that $|p'|^2, \ldots, |p^{(k)}|^2$ are functions not convex on $\mathbb{C}$.

Therefore $u_1, \ldots, u_k$ are functions not psh on $\mathbb{C}^2$. $|q^{(k)}|^2$ is not convex on $\mathbb{C}$, $|q'|^2, \ldots, |q^{(k)}|^2$ are not convex functions on $\mathbb{C}$.

It follows that $v_1, \ldots, v_k$ are not psh functions on $\mathbb{C}^2$. Note that $\varphi_1(0, w) = \psi_1(w) = |p'(w)|^2 + |q'(w)|^2 = 2(|p_1'(w)|^2 + A^2)$, for $w \in \mathbb{C}$. Since $|p_1'|^2$ is not convex on $\mathbb{C}$, ψ_1 is not convex on $\mathbb{C}$. Therefore φ_1 is not psh on $\mathbb{C}^2$. $\varphi_2(0, w) = \psi_2(w) = |p''(w)|^2 + |q''(w)|^2 = 2|p_1''(w)|^2$, for $w \in \mathbb{C}$. ψ_2 is then not convex on $\mathbb{C}$. Therefore φ_2 is not psh on $\mathbb{C}^2$. Observe that $\varphi_k(0, w) = \psi_k(w) = |p^{(k)}(w)|^2 + |q^{(k)}(w)|^2 = 2|p_1^{(k)}(w)|^2$, for $w \in \mathbb{C}$.

Thus ψ_k is not convex on $\mathbb{C}$. Therefore φ_k is not psh on $\mathbb{C}^2$. □

Observe that if p, q are analytic polynomials on $\mathbb{C}$ and K is psh on $\mathbb{C}^2$, then K_1 is psh on $\mathbb{C}^2$. But F is psh on $\mathbb{C}^2$, does not implies that F_1 is psh on $\mathbb{C}^2$. Where $K(z,w) = |p(w-\overline{z})|^2$, $K_1(z,w) = |p'(w-\overline{z})|^2$, $F(z,w) = (|p(w-\overline{z})|^2 + |q(w-\overline{z})|^2)$, $F_1(z,w) = (|p'(w-\overline{z})|^2 + |q'(w-\overline{z})|^2)$, $(z,w) \in \mathbb{C}^2$.

We have the following.

Theorem 2.15. *Let $n \in \mathbb{N}$, $n \geq 3$. There exists an infinite number of analytic polynomials p, q on $\mathbb{C}$ such that if we define $u(z,w) = |p(w-\overline{z})|^2$, $v(z,w) = |q(w-\overline{z})|^2$, $u_j(z,w) = |p^{(j)}(w-\overline{z})|^2$, $v_j(z,w) = |q^{(j)}(w-\overline{z})|^2$, $j \in \{1, ..., n-2\}$, $(z,w) \in \mathbb{C}^2$. Define $\varphi(z,w) = (|p(w-\overline{z})|^2 + |q(w-\overline{z})|^2)$, $\varphi_j(z,w) = (|p^{(j)}(w-\overline{z})|^2 + |q^{(j)}(w-\overline{z})|^2)$, for $(z,w) \in \mathbb{C}^2$ and $j \in \{1, \ldots, n-2\}$, then $u, u_1, \ldots, u_{n-2}, v, v_1, \ldots, v_{n-2}$ are not psh functions on $\mathbb{C}^2$ and φ is psh on $\mathbb{C}^2$, but $\varphi_1, \ldots, \varphi_{n-2}$ are not psh functions on $\mathbb{C}^2$.*

Proof. Let $p_1(w) = w^n - w^{n-1}$, for $w \in \mathbb{C}$. p_1 is a analytic polynomial on $\mathbb{C}$. Then $|p_1(w) + \alpha w|$ is not convex on $\mathbb{C}$, for all $\alpha \in \mathbb{C}$. Observe that $|p_1|^2, \ldots, |p_1^{(n-2)}|^2$ are not convex functions on $\mathbb{C}$. Define $p(w) = p_1(w) + Aw$, $q(w) = p_1(w) - Aw$ for $w \in \mathbb{C}$, where $A > 0$, A to be constructed satisfying the following hypotheses proven below. Now

$$\lim_{|w| \to +\infty} \frac{|p_1''(w)p_1(w)|}{|p_1'(w)|^2} = \frac{n-1}{n} < 1.$$

Thus there exists $B > 0$ such that $|w| > B$ so that $|p_1''(w)p_1(w)| < |p_1'(w)|^2$. If B is fixed, then $\overline{D}(0,B)$ is a compact subset of $\mathbb{C}$ and the function $|p_1''p_1|$ is continuous on $\overline{D}(0,B)$. Therefore there exists $A_1 > 0$ such that $|p_1''(w)p_1(w)| < A_1^2$, for all $w \in \overline{D}(0,B)$. We conclude that $|p_1''(w)p_1(w)| < A_1^2 + |p_1'(w)|^2$, for every $w \in \mathbb{C}$. Since $\{\alpha \in \mathbb{C} \ / \ |p_1' + \alpha|^2 \text{ is convex on } \mathbb{C}\}$ have a cardinal less than 1, there exists $A_2 \geq A_1$, such that for all $\alpha \in \mathbb{C}$, with $|\alpha| \geq A_2$, the function $|p_1' + \alpha|^2$ is not convex on $\mathbb{C}$. Now let $A \in \mathbb{R}_+$, $A \geq A_2$. We have $(|p|^2 + |q|^2)$ is then convex on $\mathbb{C}$. $(|p'|^2 + |q'|^2) = 2(|p_1'|^2 + A^2)$ is not convex on $\mathbb{C}$. In fact the function $(|p^{(j)}|^2 + |q^{(j)}|^2)$ is not convex on $\mathbb{C}$, for all $j \in \{1, \ldots, n-2\}$. Note that $|p^{(j)}|^2$, $|q^{(j)}|^2$ are not convex functions on $\mathbb{C}$, for all $j \in \{1, \ldots, n-2\}$. □

Theorem 2.16. *Let $n, N \in \mathbb{N}$, $n \geq 3$ and $N \geq 2$. There exists an infinite number of analytic polynomials p, q on $\mathbb{C}^N$, $deg(p) = deg(q) = n$, such that $|p|^2$ and $|q|^2$ are not convex functions on $\mathbb{C}^N$, $(|p|^2 + |q|^2)$ is convex on $\mathbb{C}^N$ and $\frac{\partial^2}{\partial z_j \partial \overline{z}_j}(|p|^2 + |q|^2)$ is not convex on $\mathbb{C}^N$, for all $j \in \{1, \ldots, N\}$.*

Proof. Let p_1 and q_1 be 2 analytic polynomials on $\mathbb{C}$, $\deg(p_1) = \deg(q_1) = n$, such that $|p_1|^2$ and $|q_1|^2$ are not convex functions on $\mathbb{C}$, $(|p_1|^2 + |q_1|^2)$ is convex on $\mathbb{C}$ and $(|p_1'|^2 + |q_1'|^2)$ is not convex on $\mathbb{C}$. Let $a = (a_1, \ldots, a_N) \in (\mathbb{C} \backslash \{0\})^N$. Define p and q on $\mathbb{C}^N$ by $p(z) = p_1(< z/a >)$ and $q(z) = q_1(< z/a >)$, for all $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$. Then p and q are holomorphic polynomials on $\mathbb{C}^N$, $\deg(p) = \deg(q) = n$. Indeed, p and q satisfy the condition of the theorem. □

Remark 2.17. In fact we have for all analytic polynomials p on $\mathbb{C}$, there exists always $A \in \mathbb{C}$ such that u is psh (or strictly psh) on $\mathbb{C}^2$, where $u(z,w) = |p(w-\overline{z})|^2 + |A(w-\overline{z})|^2$, for $(z,w) \in \mathbb{C}^2$. But this property is not true in general for analytic functions on $\mathbb{C}$.

Example 2.18. Let $g(z) = e^{(z^2)}$, for $z \in \mathbb{C}$. Let $A \in \mathbb{C}$ and define

$$v(z,w) = |g(w-\overline{z})|^2 + |A(w-\overline{z})|^2,$$

$(z,w) \in \mathbb{C}^2$. Then v is not psh on $\mathbb{C}^2$, because if $|g''(z)\overline{g}(z)| \leq |g'(z)|^2 + |A|^2$, for all $z \in \mathbb{C}$, then we have $(2 + 4x^2)e^{2x^2} \leq 4x^2 e^{2x^2} + |A|^2$, for any $x \in \mathbb{R}$.

Therefore $2e^{2x^2} \leq |A|^2$, for any $x \in \mathbb{R}$. We have a contradiction. On the other hand, on $\mathbb{C}^n$, $n \geq 2$, the above property is not true for analytic polynomials. This is one of the great differences between the theory of functions of one complex variable and the same theory in several variables. Exactly, there exists an analytic polynomial q on $\mathbb{C}^n$ such that for all $A \in \mathbb{C}$, the inequality

$$|\sum_{j,k=1}^{n} \frac{\partial^2 q}{\partial z_j \partial z_k}(z)\overline{q}(z)\alpha_j\alpha_k| \leq |\sum_{j=1}^{n} \frac{\partial q}{\partial z_j}(z)\alpha_j|^2 + |A|^2 \parallel \alpha \parallel^2$$

for every $z = (z_1, ..., z_n) \in \mathbb{C}^n$, for each $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$ is impossible if $n \geq 2$. The answer is given by the following proposition.

Proposition 2.19. *Let $q(z_1, z_2) = z_1 z_2$, $(z_1, z_2) \in \mathbb{C}^2$, where q is an analytic polynomial on $\mathbb{C}^2$. There does not exist a constant $A \in \mathbb{C}$, such that u is convex on $\mathbb{C}^2$, $u(z_1, z_2) = |z_1 z_2|^2 + \parallel A(z_1, z_2) \parallel^2$, for $(z_1, z_2) \in \mathbb{C}^2$.*

Proof. Assume that there exists $A \in \mathbb{C}$ such that u is convex on $\mathbb{C}^2$. Then $|2\overline{q}(z)\alpha_1\alpha_2| \leq |z_2\alpha_1 + z_1\alpha_2|^2 + |A|^2 \parallel \alpha \parallel^2$, for every $z = (z_1, z_2) \in \mathbb{C}^2$, for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2$. Put $z_1 = \alpha_1$, $z_2 = -\alpha_2 \in \mathbb{C}$. Then we have $|2\alpha_1^2\alpha_2^2| \leq |A|^2(|\alpha_1|^2 + |\alpha_2|^2)$, for all $(\alpha_1, \alpha_2) \in \mathbb{C}^2$. Put now $\alpha_2 = \alpha_1 \in \mathbb{C}\backslash\{0\}$. Thus $|\alpha_1|^2 \leq |A|^2$, for every $\alpha_1 \in \mathbb{C}\backslash\{0\}$. It follows that we have a contradiction. Finally we can study the convexity of the function $\parallel f \parallel$, where $f : \mathbb{C}^n \to \mathbb{C}^N$ is holomorphic, $f = (f_1, ..., f_N)$, $n, N \geq 1$. □

3 Holomorphic functions and the real convexity

Theorem 3.1. *Let $g : \mathbb{C}^n \to \mathbb{C}$ be analytic and $|g| > 0$ on $\mathbb{C}^n$, $n \geq 1$. Suppose that $|g|$ is convex on $\mathbb{C}^n$. Then $g(z) = e^{F(z)}$, for all $z \in \mathbb{C}^n$, where $F : \mathbb{C}^n \to \mathbb{C}$ is analytic and affine on $\mathbb{C}^n$.*

Proof. The proof is by induction on $n \geq 1$. If $n = 1$, by Abidi [2], we have $g(z) = e^{(az+b)}$, for all $z \in \mathbb{C}$, where $a, b \in \mathbb{C}$.

If $n = 2$. Since $|g| > 0$ on $\mathbb{C}^2$, then $g(z) = e^{F(z)}$, for all $z \in \mathbb{C}^2$, where $F : \mathbb{C}^2 \to \mathbb{C}$, F analytic on $\mathbb{C}^2$. For $z_2 \in \mathbb{C}$, the function $g(., z_2) = e^{F(., z_2)}$ is analytic on $\mathbb{C}$ and $|g(., z_2)|$ is convex on $\mathbb{C}$. Therefore by [2, Theorem 20], $F(z_1, z_2) = c(z_2)z_1 + d(z_2)$, for each $z_1 \in \mathbb{C}$, with $c, d : \mathbb{C} \to \mathbb{C}$ and where $d(z_2) = F(0, z_2)$ and $c(z_2) = F(1, z_2) - d(z_2)$, for all $z_2 \in \mathbb{C}$. Therefore c and d are analytic functions on $\mathbb{C}$ and $g(z_1, z_2) = e^{(c(z_2)z_1 + d(z_2))}$, for any $(z_1, z_2) \in \mathbb{C}^2$.

Now $g(0, z_2) = e^{d(z_2)}$, $g(0, .)$ is holomorphic on $\mathbb{C}$, $|g(0, .)|$ is convex on $\mathbb{C}$. By Abidi [2], d is an affine function on $\mathbb{C}$. On the other hand, $g(1, z_2) = e^{(c(z_2)+d(z_2))}$, for all $z_2 \in \mathbb{C}$. But $g(1, .)$ is holomorphic and $|g(1, .)|$ is convex on $\mathbb{C}$. Then the function $(c + d)$ is a holomorphic affine function on $\mathbb{C}$. Since now $c = (c + d) - d$ on $\mathbb{C}$, c is a holomorphic affine function on $\mathbb{C}$. $c(z_2) = c_1 z_2 + c_2$ and $d(z_2) = d_1 z_2 + d_2$, for all $z_2 \in \mathbb{C}$, where $c_1, c_2, d_1, d_2 \in \mathbb{C}$. $g(z_1, z_2) = e^{((c_1 z_2 + c_2)z_1 + d_1 z_2 + d_2)} = e^{(c_1 z_1 z_2 + c_2 z_1 + d_1 z_2 + d_2)}$, for all $(z_1, z_2) \in \mathbb{C}^2$.

We will prove that $c_1 = 0$. Put $z_2 = z_1$. Define $\varphi(z_1) = g(z_1, z_1) = e^{(c_1 z_1^2 + (c_2 + d_1)z_1 + d_2)}$, for $z_1 \in \mathbb{C}$. Then φ is holomorphic and $|\varphi|$ is convex on $\mathbb{C}$. From [2], we get that $c_1 = 0$.

It follows that $g(z_1, z_2) = e^{(c_2 z_1 + d_1 z_2 + d_2)}$, for all $z = (z_1, z_2) \in \mathbb{C}^2$. Now assume that for all $g_1 : \mathbb{C}^n \to \mathbb{C}$, g_1 is analytic, $|g_1| > 0$ and $|g_1|$ is convex on $\mathbb{C}^n$, then $g_1(z) = e^{F_1(z)}$, where F_1 is holomorphic and affine on $\mathbb{C}^n$, $(n \geq 2)$.

Let now $g : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function, $|g| > 0$ and $|g|$ is convex on $\mathbb{C}^{n+1}$. Then $g(z) = e^{F(z)}$, where $F : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic function. Let $Z_1 = (z_2, ..., z_{n+1}) \in \mathbb{C}^n$. For all $z_1 \in \mathbb{C}$, we have $g(z_1, Z_1) = e^{F(z_1, Z_1)}$ and $|g(., Z_1)|$ is convex on $\mathbb{C}$. Then $F(z_1, Z_1) = c(Z_1)z_1 + d(Z_1)$, for all $z_1 \in \mathbb{C}$, with $d(Z_1) = F(0, Z_1)$ and $c(Z_1) + d(Z_1) = F(1, Z_1)$. Hence $c(Z_1) = F(1, Z_1) - F(0, Z_1)$.

Therefore c and d are holomorphic functions on $\mathbb{C}^n$.

Note that $g(0, Z_1) = e^{d(Z_1)}$, for $Z_1 \in \mathbb{C}^n$. Thus $g(0, .)$ is holomorphic on $\mathbb{C}^n$ and $|g(0, .)|$ is convex on $\mathbb{C}^n$. Then $d(Z_1) = d_2 z_2 + ... + d_{n+1} z_{n+1} + d_{n+2}$, for all $Z_1 = (z_2, ..., z_{n+1}) \in \mathbb{C}^n$, where $d_2, ..., d_{n+1}, d_{n+2} \in \mathbb{C}$. Since $g(1, Z_1) = e^{(c(Z_1)+d(Z_1))}$, for all $Z_1 = (z_2, ..., z_{n+1}) \in \mathbb{C}^n$ and $|g(1, .)|$ is convex on $\mathbb{C}^n$, $(c + d)$ is affine on $\mathbb{C}^n$. Now since d is affine on $\mathbb{C}^n$, c is affine on $\mathbb{C}^n$. Write $c(Z_1) = c_2 z_2 + ... + c_{n+1} z_{n+1} + c_{n+2}$, $c_2, ..., c_{n+1}, c_{n+2} \in \mathbb{C}$.

Therefore

$$\begin{aligned} g(z) = g(z_1, Z_1) &= e^{((c_2 z_2 + ... + c_{n+1} z_{n+1} + c_{n+2}) z_1 + d(Z_1))} \\ &= e^{(c_2 z_1 z_2 + ... + c_{n+1} z_1 z_{n+1} + c_{n+2} z_1 + d_2 z_2 + ... + d_{n+1} z_{n+1} + d_{n+2})}, \end{aligned}$$

$z = (z_1, Z_1) \in \mathbb{C} \times \mathbb{C}^n$.

We will prove that $c_2 = ... = c_{n+1} = 0$. Fix $(z_3^0, ..., z_{n+1}^0) \in \mathbb{C}^{n-1}$. We have

$$g(z_1, z_2, z_3^0, ..., z_{n+1}^0) = e^{(c_2 z_1 z_2 + \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3)},$$

for all $(z_1, z_2) \in \mathbb{C}^2$, where c_2, λ_1, λ_2, $\lambda_3 \in \mathbb{C}$. Since $|g(., ., z_3^0, ..., z_{n+1}^0)|$ is convex on $\mathbb{C}^2$, $c_2 = 0$ by the hypothesis of induction. It follows that $c_3 = ... = c_{n+1} = 0$. Consequently, F is affine on $\mathbb{C}^{n+1}$. The proof is now finished. □

Corollary 3.2. *Let $g : \mathbb{C}^n \to \mathbb{C}$ be an analytic function, $n \geq 1$. Define $F_k = \exp o \exp o \ldots o \exp$ $(k-$ times$)$, where $k \in \mathbb{N}$, $k \geq 2$. Assume that $|F_k(g)| = u$ is convex on $\mathbb{C}^n$. Then g is constant on $\mathbb{C}^n$.*

Corollary 3.3. *Let $g : \mathbb{C}^n \to \mathbb{C}$, g analytic, $n \geq 1$. Recall that $|e^g e^{(-g)}|$ is convex on $\mathbb{C}^n$. But we have $|e^{e^g} e^{e^{(-g)}}|$ is convex on $\mathbb{C}^n$ if and only if g is constant on $\mathbb{C}^n$. Denote by $F_k = \exp o \exp o \ldots o \exp$ $(k-$ times$)$, where $k \in \mathbb{N}$, $k \geq 2$. Assume that $|F_k(g) F_k(-g)|$ is convex on $\mathbb{C}^n$. Then g is constant on $\mathbb{C}^n$.*

Proof. Case 1. $n = 1$.

Suppose that $|e^{e^g} e^{e^{(-g)}}| = |e^{(e^g + e^{(-g)})}|$ is convex on $\mathbb{C}$. Then $|e^g + e^{(-g)}|$ is an affine function on $\mathbb{C}$. By the Picard theorem, for all $j \in \mathbb{N} \backslash \{0\}$, there exist $A_j > j$, $\exists z_j \in \mathbb{C}$, with $j < |z_j| < A_j$ and $g(z_j) \in i\mathbb{R}$. Thus $|e^{g(z_j)} + e^{-g(z_j)}| \leq |e^{g(z_j)}| + |e^{-g(z_j)}| = 2$. Since now $e^{g(z)} + e^{-g(z)} = az + b$, for any $z \in \mathbb{C}$ $(a, b \in \mathbb{C})$, $|e^{g(z_j)} + e^{-g(z_j)}| = |az_j + b| \leq 2$, for all $j \geq 1$.

But $\lim\limits_{j \to +\infty} |z_j| = +\infty$. It follows that $2 \geq \lim\limits_{j \to +\infty} |az_j + b| \geq \lim\limits_{j \to +\infty} (|a||z_j| - |b|) = +\infty$, if $a \neq 0$. This is a contradiction. Consequently, $a = 0$ and $e^g + e^{-g} = b$ on $\mathbb{C}$. The derivative relative to z implies $g'(z) e^{g(z)} - g'(z) e^{-g(z)} = 0$, for all $z \in \mathbb{C}$. Then $g'(z)(e^{2g(z)} - 1) = 0$, for any $z \in \mathbb{C}$. Since g' and $(e^{2g} - 1)$ are analytic functions, then $g' = 0$ or $(e^{2g} - 1) = 0$ on $\mathbb{C}$.

If $g' = 0$ on $\mathbb{C}$, then g is constant on $\mathbb{C}$. Now if $e^{2g} - 1 = 0$ on $\mathbb{C}$, then the derivative relative to z implies that $g' e^{2g} = 0$ on $\mathbb{C}$ and therefore $g' = 0$ in $\mathbb{C}$. Consequently, g is constant on $\mathbb{C}$.

Case 2. $n \geq 2$.

The case is obvious by the problem of fibration. □

Corollary 3.4. *Let $g : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function, $n \geq 1$. Let*

$$u = |F_2(g) F_2(-g) F_3(g) F_3(-g) F_4(g) F_4(-g)|,$$

where $F_k = \exp o \exp o \ldots o \exp$ $(k-$ times$)$, for $k \in \mathbb{N}$, $k \geq 2$. Define $v(z, w) = u(w - \overline{z})$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. Suppose that v is psh on $\mathbb{C}^n \times \mathbb{C}^n$. Then g is constant on $\mathbb{C}^n$.

Corollary 3.5. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions, $n \geq 1$.*
(I) Assume that $|e^{g_1} + e^{g_2}|$ is a convex function on $\mathbb{C}^n$. We can not conclude that g_1 and g_2 are constant in $\mathbb{C}^n$. Moreover,
(II) Assume that $|e^{e^{g_1}} + e^{e^{g_2}}|$ is convex on $\mathbb{C}^n$. Then g_1 and g_2 are constant in $\mathbb{C}^n$.

Proof. (I). Let $g_2 = g_1 + i\pi$ and g_1 is a function nonconstant on $\mathbb{C}^n$. Then $|e^{g_1} + e^{g_2}| = 0$ is convex on $\mathbb{C}^n$.

(II). In fact in general we prove that if $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ are 2 analytic functions such that $|e^{e^{f_1}} + e^{f_2}|$ is convex on $\mathbb{C}^n$, then f_1 and f_2 are constant on $\mathbb{C}^n$. □

Observation 3.6. We can use the above theorem for the resolution of several holomorphic partial differential equations on $\mathbb{C}^n$, $n \geq 1$. For example find all the holomorphic functions $g : \mathbb{C} \to \mathbb{C}$ such that $u_1 > 0$ and u_1 is convex on $\mathbb{C}$, $u_1(z) = |ag'(z) + bg^{(3)}(z)|^3$, for $z \in \mathbb{C}$ ($a, b \in \mathbb{C}\backslash\{0\}$). Find all the holomorphic functions $k : \mathbb{C}^n \to \mathbb{C}$ such that $u_2 > 0$ and u_2 is convex on $\mathbb{C}^n$, $u_2(z) = |k(z) + \frac{\partial k}{\partial z_1}(z)|^4$, for $z = (z_1, ..., z_n) \in \mathbb{C}^n$. Find all the holomorphic functions $k : \mathbb{C}^n \to \mathbb{C}$ such that $u_3 > 0$ and u_3 is convex on $\mathbb{C}^n$, where $u_3(z) = |\frac{\partial^2 k}{\partial z_1^2}(z) + ... + \frac{\partial^2 k}{\partial z_n^2}(z)|^6$, for $z = (z_1, ..., z_n) \in \mathbb{C}^n$.

Theorem 3.7. *Let $g : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function, $n \geq 1$. Assume that $|g|$ is convex on $\mathbb{C}^n$ and $g(z^0) = 0$, where $z^0 \in \mathbb{C}^n$. Then $g(z) = (< z/\lambda > +\mu)^m$, for all $z \in \mathbb{C}^n$, where $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$ and $m \in \mathbb{N}$.*

Proof. The proof is by induction on $n \geq 1$.

If $n = 1$. From Abidi [2], we have the proof.
Suppose that $n = 2$. We assume that $z^0 = 0$ (if $z^0 \neq 0$, we consider the function k defined in $\mathbb{C}^2$ by $k(z) = g(z + z^0)$, $z \in \mathbb{C}^2$). If $g(z_1, z_2) = g_1(z_1)$, for any $(z_1, z_2) \in \mathbb{C}^2$, the theorem is true ($g_1 : \mathbb{C} \to \mathbb{C}, g_1$ is holomorphic on $\mathbb{C}$). Now suppose that $g(z_1, z_2)$ depends on z_1 and z_2. Then $\frac{\partial g}{\partial z_1}\frac{\partial g}{\partial z_2} \neq 0$ on $\mathbb{C}^2$.

If $g(z_1, 0) = 0$ and $g(0, z_2) = 0$, for each $(z_1, z_2) \in \mathbb{C}^2$, then we have

$$|g(\frac{z_1}{2}, \frac{z_2}{2})| = |g(\frac{1}{2}(z_1, 0) + \frac{1}{2}(0, z_2))| \leq \frac{1}{2}|g(z_1, 0)| + \frac{1}{2}|g(0, z_2)| = 0.$$

Then $g(\frac{z_1}{2}, \frac{z_2}{2}) = 0$, for all $(z_1, z_2) \in \mathbb{C}^2$. Consequently, $g = 0$ on $\mathbb{C}^2$. This is a contradiction.

Now in fact we have $g(., 0) \neq 0$ on $\mathbb{C}$ if $g(z_1, 0) = 0$, for every $z_1 \in \mathbb{C}$. We conclude by the same above proof that $\frac{\partial g}{\partial z_1}(z_1, z_2) = 0$, for any $(z_1, z_2) \in \mathbb{C}^2$. This is impossible because $\frac{\partial g}{\partial z_1}\frac{\partial g}{\partial z_2} \neq 0$ on $\mathbb{C}^2$. We have $g(z_1, 0) = (a(0)z_1 + b(0))^s$, for every $z_1 \in \mathbb{C}$, where $a(0) \in \mathbb{C}$, $b(0) \in \mathbb{C}$ and $s \in \mathbb{N}$. Since $g(0, 0) = 0$, $b(0) = 0$ and $s \in \mathbb{N}\backslash\{0\}$.

Suppose now that $g(z_1, \xi_j) = e^{(\lambda_j z_1 + \mu_j)}$, for each $z_1 \in \mathbb{C}$, where $\lambda_j, \mu_j \in \mathbb{C}$, for every $j \in \mathbb{N}\backslash\{0\}$ and the sequence $(\xi_j)_{j \geq 1} \subset D(0, r)$, $r > 0$, $\lim_{j \to +\infty}(\xi_j) = 0$.

Let $z_1 \in \mathbb{C}\backslash\{0\}$. By [2], we have

$$\frac{\partial^2 g}{\partial z_1^2}(z_1, 0)g(z_1, 0) = \frac{s-1}{s}(\frac{\partial g}{\partial z_1}(z_1, 0))^2,$$

and

$$\frac{\partial^2 g}{\partial z_1^2}(z_1, \xi_j)g(z_1, \xi_j) = (\frac{\partial g}{\partial z_1}(z_1, \xi_j))^2,$$

for each $j \in \mathbb{N}\backslash\{0\}$. Then we have $\lim_{j \to +\infty}(\frac{\partial g}{\partial z_1}(z_1, \xi_j))^2 = \frac{s-1}{s}(\frac{\partial g}{\partial z_1}(z_1, 0))^2$.
Hence $a(0) = 0$ and $0 = g(z_1, 0)$, for each $z_1 \in \mathbb{C}$. This is a contradiction.

Thus there exists $r > 0$ such that $g(z_1, z_2) = (a(z_2)z_1 + b(z_2))^{s_1(z_2)}$, for every $z_1 \in \mathbb{C}$, for any $z_2 \in D(0, r)$, where $a(z_2), b(z_2) \in \mathbb{C}$, $s_1(z_2) \in \mathbb{N}$ and the function $z_1 \in \mathbb{C} \mapsto (a(z_2)z_1 + b(z_2))^{s_1(z_2)}$ is noconstant, for all $z_2 \in D(0, r)$. In fact if there exists a sequence $(z_{2,j})_{j\geq 1} \subset \mathbb{C}$, with $\lim\limits_{j\to+\infty} z_{2,j} = 0$ and the function $g(., z_{2,j}) = c_j$ in $\mathbb{C}$, where $c_j \in \mathbb{C}$. For $z_1 \in \mathbb{C}$, we have $\lim\limits_{j\to+\infty} g(z_1, z_{2,j}) = g(z_1, 0)$. Then $\lim\limits_{j\to+\infty}(c_j) = c \in \mathbb{C}$. Now since $g(z_1, z_{2,j}) = g(z_1', z_{2,j})$, for any $z_1' \in \mathbb{C}$, then c is independent of $z_1 \in \mathbb{C}$. Thus $g(0, 0) = c = 0$ and then $g(z_1, 0) = 0$, for every $z_1 \in \mathbb{C}$, a contradiction. Then $s_1(z_2) \in \mathbb{N}\backslash\{0\}$, for each $z_2 \in D(0, r)$. Let $z_2 \in D(0, r)$. We have

$$\frac{\partial^2 g}{\partial z_1^2}(z_1, z_2')g(z_1, z_2') = \frac{s_1(z_2') - 1}{s_1(z_2')}\left(\frac{\partial g}{\partial z_1}(z_1, z_2')\right)^2,$$

for each $z_2' \in D(0, r)$, $z_1 \in \mathbb{C}$. $\frac{\partial^2 g}{\partial z_1^2}(z_1, z_2)g(z_1, z_2) = \frac{s_1(z_2)-1}{s_1(z_2)}(\frac{\partial g}{\partial z_1}(z_1, z_2))^2$. Then

$$\lim_{z_2'\to z_2}(1 - \frac{1}{s_1(z_2')}) = 1 - \frac{1}{s_1(z_2)}.$$

Thus $\lim\limits_{z_2'\to z_2} s_1(z_2') = s_1(z_2)$, for each $z_2 \in D(0, r)$. This implies that $s_1 : D(0, r) \to \mathbb{N}\backslash\{0\}$ is a continuous function. It follows that s_1 is constant on $D(0, r)$. Let $s_1(z_2) = s$, for every $z_2 \in D(0, r)$. Now $g(z_1, z_2) = (a(z_2)z_1 + b(z_2))^s$, for every $z_1 \in \mathbb{C}$, $z_2 \in D(0, r)$. Now choose $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2$, $\rho > 0$, such that $D(\alpha_1, \rho) \times D(\alpha_2, \rho) \subset D(0, r) \times D(0, r)$ and

$$\begin{cases} g(z) \neq 0, \text{ for } z \in D(\alpha_1, \rho) \times D(\alpha_2, \rho); \\ \frac{\partial g}{\partial z_1}(z)\frac{\partial g}{\partial z_2}(z) \neq 0, \text{ for } z \in D(\alpha_1, \rho) \times D(\alpha_2, \rho); \\ g(z_1, 0) \neq 0, g(0, z_2) \neq 0, \text{ for } (z_1, z_2) \in D(\alpha_1, \rho) \times D(\alpha_2, \rho). \end{cases}$$

On the other hand, $g(0, z_2) = (c(0)z_2 + d(0))^q$, for any $z_2 \in \mathbb{C}$, with $c(0), d(0) \in \mathbb{C}$, $q \in \mathbb{N}\backslash\{0\}$.

If $g(0, z_2) = 0$, for all $z_2 \in \mathbb{C}$ ($c(0) = d(0) = 0$ and $q \neq 0$). This is impossible because we have the inequality $|g(\frac{z_1}{2}, \frac{z_2}{2})| = |g((\frac{z_1}{2}, 0) + (0, \frac{z_2}{2}))| = |g(\frac{1}{2}(z_1, 0) + \frac{1}{2}(0, z_2))| \leq \frac{1}{2}|g(z_1, 0)| + \frac{1}{2}|g(0, z_2)| = \frac{1}{2}|g(z_1, 0)|$, for all $(z_1, z_2) \in \mathbb{C}^2$. Then $|g(\frac{z_1}{2}, \frac{z_2}{2})| \leq \frac{1}{2}|g(z_1, 0)|$, for all $z_1, z_2 \in \mathbb{C}$ and $|g(z_1, z_2)| \leq \frac{1}{2}|g(2z_1, 0)|$, for any $(z_1, z_2) \in \mathbb{C}^2$. Hence $g(z_1, z_2) = \gamma g(2z_1, 0)$, for each $(z_1, z_2) \in \mathbb{C}^2$, where $\gamma \in \mathbb{C}$. Thus $\frac{\partial g}{\partial z_1}(z_1, z_2)\frac{\partial g}{\partial z_2}(z_1, z_2) = 0$, for every $(z_1, z_2) \in \mathbb{C}^2$. This is a contradiction. Therefore $g(0, .) \neq 0$ on $\mathbb{C}$.

Now if $g(\zeta_j, z_2) = e^{(\gamma_j z_2 + \delta_j)}$, for every $z_2 \in \mathbb{C}$, where $\gamma_j, \delta_j \in \mathbb{C}$, for all $j \in \mathbb{N}\backslash\{0\}$ and the sequence $(\zeta_j)_{j\geq 1} \subset D(0, r)$, with $\lim\limits_{j\to+\infty} \zeta_j = 0$. We have $\frac{\partial^2 g}{\partial z_2^2}(0, z_2)g(0, z_2) = \frac{q-1}{q}(\frac{\partial g}{\partial z_2}(0, z_2))^2$ and $\frac{\partial^2 g}{\partial z_2^2}(\zeta_j, z_2)g(\zeta_j, z_2) = (\frac{\partial g}{\partial z_2}(\zeta_j, z_2))^2$. Then $\frac{\partial g}{\partial z_2}(0, z_2) = 0$, for each $z_2 \in \mathbb{C}$ and $c(0) = 0$. Therefore $g(0, z_2) = 0 = g(0, 0)$, for each $z_2 \in \mathbb{C}$. This is a contradiction.

Consequently, $g(z_1, z_2) = (c(z_1)z_2 + d(z_1))^{q_1(z_1)}$, for every $(z_1, z_2) \in D(0, \rho_1) \times D(0, \rho_1)$, $g(z_1, .)$ is nonconstant on $D(0, \rho_1)$, for all $z_1 \in D(0, \rho_1)$, where $0 < \rho_1 \leq \rho$ and where $q_1(z_1) \in \mathbb{N}\backslash\{0\}$, $c, d : D(0, \rho_1) \to \mathbb{C}$. We prove that $q_1(z_1) = q$, for every $z_1 \in D(0, \rho_1)$ (by the same method developed as above).

If $g(z) \in [0, +\infty[$, for all $z \in G = D(\alpha_1, \rho_1) \times D(\alpha_2, \rho_1)$, therefore $\text{Im}(g) = 0$ in the convex domain G. Here, we can also use the function $\text{Re}(g)$. Thus g is constant on G. This is a contradiction. Therefore there exists $(z_1^0, z_2^0) \in G$ such that $g(z_1^0, z_2^0) \in \mathbb{C}\backslash[0, +\infty[$ and $\mathbb{C}\backslash[0, +\infty[$ is an open of $\mathbb{C}$. Since $g : G \to \mathbb{C}$ is a continuous function, there exists $\eta > 0$ such that $G_1 = D(z_1^0, \eta) \times D(z_0^2, \eta) \subset G$ and for all $z \in G_1$, $g(z) \in \mathbb{C}\backslash[0, +\infty[$. We have $g(z) = (a(z_2)z_1 + b(z_2))^s = (c(z_1)z_2 + d(z_1))^q$, for all $z = (z_1, z_2) \in G_1$. Thus $g^{\frac{1}{s}}(z) = \psi(z_1, z_2)(a(z_2)z_1 + b(z_2))$, for all $z = (z_1, z_2) \in G_1$, where $\psi : G_1 \to \mathbb{C}$, $\psi^s = 1$ on G_1. Since now $g(z) \neq 0$, for all $z \in G_1$, then $(a(z_2)z_1 + b(z_2)) \neq 0$, for all $z = (z_1, z_2) \in G_1$. It follows that $\psi(z_1, z_2) = \frac{g^{\frac{1}{s}}(z_1, z_2)}{a(z_2)z_1 + b(z_2)}$, for

$(z_1, z_2) \in G_1$. Fix $z_2 \in D(z_2^0, \eta)$. Observe that $(z_1 \mapsto \psi(z_1, z_2))$ is a holomorphic function on $D(z_1^0, \eta)$ and satisfy $(\psi(z_1, z_2))^s = 1$, for each $z_1 \in D(z_1^0, \eta)$.

It follows that the function $\psi(., z_2)$ is constant on $D(z_1^0, \eta)$. Define then $\varphi(z_2) = \psi(z_1, z_2)$, for every $(z_1, z_2) \in G_1$. This implies that $\varphi : D(z_2^0, \eta) \to \mathbb{C}$ is then a function. We have $g(z_1, z_2) = (\varphi(z_2)a(z_2)z_1 + \varphi(z_2)b(z_2))^s$, for each $(z_1, z_2) \in G_1$. Define $A = \varphi a$, $B = \varphi b$ in $D(z_2^0, \eta)$. We can calculate A and B on $D(z_2^0, \eta)$. In fact fix $z_1^1 \in D(z_1^0, \eta)$, $z_1^1 \neq z_1^0$. We have then

$$g^{\frac{1}{s}}(z_1^0, z_2) = A(z_2)z_1^0 + B(z_2),$$
$$g^{\frac{1}{s}}(z_1^1, z_2) = A(z_2)z_1^1 + B(z_2),$$

for all $z_2 \in D(z_2^0, \eta)$. Consequently,

$$A(z_2) = \frac{1}{(z_1^0 - z_1^1)}(g^{\frac{1}{s}}(z_1^0, z_2) - g^{\frac{1}{s}}(z_1^1, z_2)), B(z_2) = g^{\frac{1}{s}}(z_1^0, z_2) - A(z_2)z_1^0.$$

It follows that A and B are holomorphic functions on $D(z_2^0, \eta)$ and so $g(z_1, z_2) = (A(z_2)z_1 + B(z_2))^s$, for each $(z_1, z_2) \in G_1$. Similarly, $g^{\frac{1}{q}}(z_1, z_2) = (c(z_1)z_2 + d(z_1))\psi_1(z_1, z_2)$, for any $(z_1, z_2) \in G_1$, where $\psi_1 : G_1 \to \mathbb{C}$, $(\psi_1)^q = 1$ on G_1. Since $g(z) \neq 0$, for each $z \in G_1$, then $(c(z_1)z_2 + d(z_1)) \neq 0$, for every $z = (z_1, z_2) \in G_1$. Then $\psi_1(z_1, z_2) = \frac{g^{\frac{1}{q}}(z_1, z_2)}{c(z_1)z_2 + d(z_1)}$, for all $(z_1, z_2) \in G_1$.

Fix $z_1 \in D(z_1^0, \eta)$. Thus $(z_2 \in D(z_2^0, \eta) \mapsto \psi_1(z_1, z_2))$ is a holomorphic function on $D(z_2^0, \eta)$ and satisfy the equality $(\psi_1(z_1, z_2))^q = 1$, for all $z_2 \in D(z_2^0, \eta)$. It follows that the function $\psi_1(z_1, .)$ is constant on $D(z_2^0, \eta)$, for all $z_1 \in D(z_1^0, \eta)$. Define now $\varphi_1(z_1) = \psi_1(z_1, z_2)$, for any $z_1 \in D(z_1^0, \eta)$, for all $z_2 \in D(z_2^0, \eta)$. $\varphi_1 : D(z_1^0, \eta) \to \mathbb{C}$ is a function. Then we have $g(z_1, z_2) = (\varphi_1(z_1)c(z_1)z_2 + \varphi_1(z_1)d(z_1))^q$, for each $(z_1, z_2) \in G_1$. Define $c_1 = \varphi_1 c$, $d_1 = \varphi_1 d$ on the domain $D(z_1^0, \eta)$. We can calculate c_1 and d_1 as follows. Fix $z_2^1 \in D(z_2^0, \eta)$, with $z_2^1 \neq z_2^0$. Then $g^{\frac{1}{q}}(z_1, z_2^0) = c_1(z_1)z_2^0 + d_1(z_1)$, $g^{\frac{1}{q}}(z_1, z_2^1) = c_1(z_1)z_2^1 + d_1(z_1)$, for all $z_1 \in D(z_1^0, \eta)$. $c_1(z_1) = \frac{1}{(z_2^0 - z_2^1)}(g^{\frac{1}{q}}(z_1, z_2^0) - g^{\frac{1}{q}}(z_1, z_2^1))$, $d_1(z_1) = g^{\frac{1}{q}}(z_1, z_2^0) - c_1(z_1)z_2^0$.

Therefore c_1 and d_1 are holomorphic functions on $D(z_1^0, \eta)$ and $g(z_1, z_2) = (c_1(z_1)z_2 + d_1(z_1))^q$, for all $(z_1, z_2) \in G_1$. Consequently, $g(z_1, z_2) = (A(z_2)z_1 + B(z_2))^s = (c_1(z_1)z_2 + d_1(z_1))^q$, for each $(z_1, z_2) \in G_1$.

We want to prove that $q = s$. Since $\frac{\partial^{s+1} g}{\partial z_1^{s+1}}(z_1, z_2) = 0$, for each $(z_1, z_2) \in G_1$, $q \leq s$. Also $\frac{\partial^{q+1} g}{\partial z_2^{q+1}}(z_1, z_2) = 0$, for every $(z_1, z_2) \in G_1$, thus $s \leq q$. Consequently, $s = q$. Therefore, $(A(z_2)z_1 + B(z_2))^s = (c_1(z_1)z_2 + d_1(z_1))^s$, for all $(z_1, z_2) \in G_1$.

Thus $A(z_2)z_1 + B(z_2) = \lambda(z)[c_1(z_1)z_2 + d_1(z_1)]$, for every $z = (z_1, z_2) \in G_1$, where $\lambda(z) \in \mathbb{C}$, $(\lambda(z))^s = 1$. Also $\lambda : G_1 \to \mathbb{C}$, λ is holomorphic on G_1 because $(A(z_2)z_1 + B(z_2)) \neq 0$, $(c_1(z_1)z_2 + d_1(z_1)) \neq 0$, for any $(z_1, z_2) \in G_1$.

Since $\lambda^s = 1$ on G_1, then λ is constant in the domain G_1. The derivative of the expression $(A(z_2)z_1 + B(z_2))$ relative to z_1 implies that $A(z_2) = \lambda[c_1'(z_1)z_2 + d_1'(z_1)]$, for all $(z_1, z_2) \in G_1$. Hence c_1' and d_1' are constant functions on $D(z_1^0, \eta)$. $c_1(z_1) = \gamma z_1 + \beta$, $d_1(z_1) = \alpha z_1 + \delta$, for all $z_1 \in D(z_1^0, \eta)$, where $\gamma, \beta, \alpha, \delta \in \mathbb{C}$. Now $g(z_1, z_2) = [\gamma z_1 z_2 + \beta z_2 + \alpha z_1 + \delta]^s$, for every $(z_1, z_2) \in G_1$. Define $f(z_1, z_2) = [\gamma z_1 z_2 + \beta z_2 + \alpha z_1 + \delta]^s$, for $(z_1, z_2) \in \mathbb{C}^2$. Then f is analytic on $\mathbb{C}^2$ and $f = g$ on the domain G_1. Hence, $f = g$ on $\mathbb{C}^2$. Now $g(z_1, z_2) = (\gamma z_1 z_2 + \beta z_2 + \alpha z_1 + \delta)^s$, for each $(z_1, z_2) \in \mathbb{C}^2$, where $s \in \mathbb{N}\backslash\{0\}$.

Suppose that $\gamma \neq 0$. Choose $\mu \in \mathbb{C}$ such that $(\alpha + \beta + \gamma\mu)^2 - 4\gamma(\beta\mu + \delta) \neq 0$. Define $K(z_1) = g(z_1, z_1 + \mu)$, for $z_1 \in \mathbb{C}$. Then $K(z_1) = [\gamma z_1^2 + (\alpha + \beta + \gamma\mu)z_1 + \beta\mu + \delta]^s$. We have $|K_1|$ is convex on $\mathbb{C}$. But K_1 is a holomorphic polynomial having 2 distinct zeros on $\mathbb{C}$. This is a contradiction. Therefore $\gamma = 0$. Consequently, $g(z_1, z_2) = (\alpha z_1 + \beta z_2 + \delta)^s$, for all $(z_1, z_2) \in \mathbb{C}^2$.

Suppose that the result is true for all $g_1 : \mathbb{C}^n \to \mathbb{C}$, g_1 analytic, $|g_1|$ is convex on $\mathbb{C}^n$ and $g_1(z^0) = 0$, where $z^0 \in \mathbb{C}^n$, $n \geq 2$.

Now let $g : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function, $|g|$ is convex on $\mathbb{C}^{n+1}$, $g(z^0) = 0$, where $z^0 \in \mathbb{C}^{n+1}$. If there exists $j \in \{1, ..., n+1\}$, such that $g(z_1, ..., z_{n+1})$ is independent of z_j, then we use the hypothesis of induction.

We now assume that $\frac{\partial g}{\partial z_1} \cdots \frac{\partial g}{\partial z_n} \frac{\partial g}{\partial z_{n+1}} \neq 0$ on $\mathbb{C}^{n+1}$ and $z^0 = 0$. Then $g(0, Z_1) = (a_2(0)z_2 + a_3(0)z_3 + \ldots + a_{n+1}(0)z_{n+1} + a_{n+2}(0))^s$ for all $Z_1 = (z_2, \ldots, z_{n+1}) \in \mathbb{C}^n$, where $s \in \mathbb{N}$, $a_2(0), a_3(0), \ldots, a_{n+1}(0), a_{n+2}(0) \in \mathbb{C}$ by the hypothesis of induction.

If $s = 0$. Then $g(0, Z_1) = 1$, for each $Z_1 = (z_2, \ldots, z_{n+1}) \in \mathbb{C}^n$. Then $g(0,0) = 1$. This is a contradiction. Therefore $s \in \mathbb{N}\backslash\{0\}$ and $a_{n+2}(0) = 0$.

If $a_2(0) = 0$, then $g(0, z_2, z_3, ..., z_{n+1})$ is independent of z_2. This is impossible because for $z = (z_1, z_2, z_3, ..., z_{n+1}) \in \mathbb{C}^{n+1}$, we have

$$\begin{aligned}
|g(\frac{1}{2}(z_1, z_2, z_3, ..., z_{n+1}))| &= |g(\frac{1}{2}(z_1, 0) + \frac{1}{2}(0, z_2, z_3, ..., z_{n+1}))| \\
&\leq \frac{1}{2}|g(z_1, 0)| + \frac{1}{2}|g(0, z_2, z_3, ..., z_{n+1})| \\
&\leq \frac{1}{2}|g(z_1, 0)| + \frac{1}{2}|g(0, 0, z_3, ..., z_{n+1})|.
\end{aligned}$$

Thus $|g(z_1, z_2, z_3, ..., z_{n+1})| \leq \frac{1}{2}|g(2z_1, 0)| + \frac{1}{2}|g(0, 0, 2z_3, ..., 2z_{n+1})|$.

Fix $z_1, z_3, ..., z_{n+1} \in \mathbb{C}$. Then the function $(z_2 \in \mathbb{C} \mapsto |g(z_1, z_2, z_3, ..., z_{n+1})|)$ is bounded above on $\mathbb{C}$ and so this function is constant relative to $z_2 \in \mathbb{C}$. It follows that $(z_2 \in \mathbb{C} \mapsto g(z_1, z_2, z_3, ..., z_{n+1}))$ is constant on $\mathbb{C}$ for all $(z_1, z_3, ..., z_{n+1}) \in \mathbb{C}^n$ fixed. Thus

$$\frac{\partial g}{\partial z_2}(z_1, z_2, z_3, ..., z_{n+1}) = 0,$$

for each $(z_1, z_2, z_3, ..., z_{n+1}) \in \mathbb{C}^{n+1}$, a contradiction. Consequently,

$$a_2(0) \neq 0, a_3(0) \neq 0, ..., a_{n+1}(0) \neq 0.$$

Assume now that $g(\xi_j, Z_1) = e^{(<Z_1/\gamma_j> + \delta_j)}$, for each $Z_1 \in \mathbb{C}^n$, $j \in \mathbb{N}\backslash\{0\}$; where $\gamma_j \in \mathbb{C}^n$, $\delta_j \in \mathbb{C}$, and the sequence $(\xi_j)_{j\geq 1} \subset \mathbb{C}$ satisfying $\lim_{j\to+\infty} \xi_j = 0$. Let $Z_1 \in (\mathbb{C}\backslash\{0\})^n$, $Z_1 = (z_2, ..., z_{n+1})$. Then $\frac{\partial^2 g}{\partial z_2^2}(\xi_j, z_2, ..., z_{n+1}) g(\xi_j, z_2, ..., z_{n+1}) = (\frac{\partial g}{\partial z_2}(\xi_j, Z_1))^2$. Hence,

$$\frac{\partial^2 g}{\partial z_2^2}(0, Z_1) g(0, Z_1) = \frac{s-1}{s}(\frac{\partial g}{\partial z_2}(0, Z_1))^2.$$

Thus $\lim_{j\to+\infty} (\dfrac{\partial g}{\partial z_2}(\xi_j, Z_1))^2 = \dfrac{s-1}{s}(\dfrac{\partial g}{\partial z_2}(0, Z_1))^2 = (\dfrac{\partial g}{\partial z_2}(0, Z_1))^2$. This implies that, $\frac{\partial g}{\partial z_2}(0, Z_1) = 0$ and so $a_2(0) = 0$, a contradiction.

It follows that there $R_1 > 0$ such that $g(z_1, Z_1) = (a_2(z_1)z_2 + a_3(z_1)z_3 + ... + a_{n+1}(z_1)z_{n+1} + a_{n+2}(z_1))^{s_1(z_1)}$, for all $(z_1, Z_1) \in D(0, R_1) \times \mathbb{C}^n$, where $s_1(z_1) \in \mathbb{N}$ for all $z_1 \in D(0, R_1))$, $a_2, a_3, ..., a_{n+1}, a_{n+2} : D(0, R_1) \to \mathbb{C}$ and the function $Z_1 \in \mathbb{C}^n \mapsto g(z_1, Z_1)$ is nonconstant relative to $z_2, ..., z_{n+1}$, for all $z_1 \in D(0, R_1)$. Thus $s_1(z_1) \in \mathbb{N}\backslash\{0\}$, for all $z_1 \in D(0, R_1)$. Now let $z_1 \in D(0, R_1)$. Choose $Z_1 \in \mathbb{C}^n$ such that $\frac{\partial g}{\partial z_1}(z_1, Z_1) \neq 0$. Then $\frac{\partial^2 g}{\partial z_1^2}(z_1, Z_1) g(z_1, Z_1) = \frac{s_1(z_1)-1}{s_1(z_1)}(\frac{\partial g}{\partial z_1}(z_1, Z_1))^2$. For $z_1' \in D(0, R_1)$, we have $\frac{\partial^2 g}{\partial z_1^2}(z_1', Z_1) g(z_1', Z_1) = \frac{s_1(z_1')-1}{s_1(z_1')}(\frac{\partial g}{\partial z_1}(z_1', Z_1))^2$. Since $\lim_{z_1'\to z_1} \dfrac{\partial^2 g}{\partial z_1^2}(z_1', Z_1) g(z_1', Z_1) = \dfrac{\partial^2 g}{\partial z_1^2}(z_1, Z_1) g(z_1, Z_1)$, we have $\lim_{z_1'\to z_1} \dfrac{s_1(z_1') - 1}{s_1(z_1')} = \dfrac{s_1(z_1) - 1}{s_1(z_1)}$

and then $\lim_{z_1'\to z_1} s_1(z_1') = s_1(z_1)$. But $s_1 : D(0,R_1) \to \mathbb{N}\backslash\{0\}$ is continuous. Thus the function s_1 is constant, $s_1(z_1) = s$, for any $z_1 \in D(0,R_1)$. Consequently, $g(z) = (a_2(z_1)z_2 + a_3(z_1)z_3 + ... + a_{n+1}(z_1)z_{n+1} + a_{n+2}(z_1))^s$, for all $(z_1,...,z_{n+1}) \in (D(0,R_1))^{n+1}$.

Using similar arguments, there exists $R_2 > 0$ such that $g(z) = (b_1(z_2)z_1 + b_3(z_2)z_3 + ... + b_{n+1}(z_2)z_{n+1} + b_{n+2}(z_2))^t$, for any $z = (z_1, z_2, z_3, \dots, z_{n+1}) \in (D(0,R_2))^{n+1}$, where $t \in \mathbb{N}$ and $b_1, b_3, ..., b_{n+1}, b_{n+2} : D(0,R_2) \to \mathbb{C}$. Let $R = \min(R_1,R_2) > 0$. Since $\frac{\partial g}{\partial z_{n+1}}(z) \neq 0$, $s = t$. Suppose that $g : (D(0,R))^{n+1} \to [0,+\infty[$. Then the function $\mathrm{Im}(g) = 0$ on $(D(0,R))^{n+1}$. It follows that g is constant on $(D(0,R))^{n+1}$. Hence g is constant on $\mathbb{C}^{n+1}$, a contradiction. Consequently, there exists $c \in (D(0,R))^{n+1}$ such that $g(c) \notin [0,+\infty[$. Note that $\mathbb{C}\backslash[0,+\infty[$ is an open on $\mathbb{C}$. Put $c = (c_1,...,c_{n+1})$. Since $g : (D(0,R))^{n+1} \to \mathbb{C}$ is a continuous function, there exists $\eta > 0$ such that $P(c,\eta) \subset (D(0,R))^{n+1}$ and $g(z) \notin [0,+\infty[$, for all $z \in P(c,\eta)$, where $P(c,\eta) = D(c_1,\eta) \times ... \times D(c_{n+1},\eta)$. Then $g : P(c,\eta) \to \mathbb{C}\backslash[0,+\infty[$, defined by $g(z) = (a_2(z_1)z_2 + a_3(z_1)z_3 + ... + a_{n+1}(z_1)z_{n+1} + a_{n+2}(z_1))^s$, for all $z = (z_1, z_2, z_3, ..., z_{n+1}) \in P(c,\eta)$. Hence $g^{\frac{1}{s}}(z) = \psi(z_1, z_2, z_3, ..., z_{n+1})(a_2(z_1)z_2 + a_3(z_1)z_3 + ... + a_{n+1}(z_1)z_{n+1} + a_{n+2}(z_1))$, for all $z = (z_1, z_2, z_3, ..., z_{n+1}) \in P(c,\eta)$, where $\psi : P(c,\eta) \to \mathbb{C}$ is a function, $(\psi)^s = 1$ on $P(c,\eta)$. Since $g(z) \neq 0$, for each $z = (z_1, z_2, z_3, ..., z_{n+1}) \in P(c,\eta)$,

$$\psi(z_1, z_2, z_3, ..., z_{n+1}) = \frac{g^{\frac{1}{s}}(z_1, z_2, z_3, ..., z_{n+1})}{(a_2(z_1)z_2 + a_3(z_1)z_3 + ... + a_{n+1}(z_1)z_{n+1} + a_{n+2}(z_1))}.$$

Fix $z_1 \in D(c_1,\eta)$. Then $\psi(z_1,.)$ is a holomorphic function in the complex variable $(z_2,...,z_{n+1}) \in D(c_2,\eta) \times ... \times D(c_{n+1},\eta)$. Since $(\psi)^s = 1$ on $P(c,\eta)$, $\psi(z_1,.)$ is constant in the open polydisc $D(c_2,\eta) \times ... \times D(c_{n+1},\eta)$. Therefore $\psi(z_1, z_2, ..., z_{n+1}) = \varphi(z_1)$, for all $(z_2,...,z_{n+1}) \in D(c_2,\eta) \times ... \times D(c_{n+1},\eta)$, for all $z_1 \in D(c_1,\eta)$, where $\varphi : D(c_1,\eta) \to \mathbb{C}$. Now we have $g^{\frac{1}{s}}(z_1, z_2, z_3, ..., z_{n+1}) = (\varphi(z_1)a_2(z_1)z_2 + \varphi(z_1)a_3(z_1)z_3 + ... + \varphi(z_1)a_{n+1}(z_1)z_{n+1} + \varphi(z_1)a_{n+2}(z_1))$, for each $(z_1, z_2, z_3, ..., z_{n+1}) \in P(c,\eta)$. Similarly, we have $g(z) = (b_1(z_2)z_1 + b_3(z_2)z_3 + ... + b_{n+1}(z_2)z_{n+1} + b_{n+2}(z_2))^s$, for each $z = (z_1, z_2, z_3, ..., z_{n+1}) \in P(c,\eta)$. Then

$$g^{\frac{1}{s}}(z) = \psi_1(z_1, z_2, z_3, ..., z_{n+1})(b_1(z_2)z_1 + b_3(z_2)z_3 + ... + b_{n+1}(z_2)z_{n+1} + b_{n+2}(z_2)),$$

for each $(z_1, z_2, z_3, ..., z_{n+1}) = z \in P(c,\eta)$, where $\psi_1 : P(c,\eta) \to \mathbb{C}$ is a function, $(\psi_1)^s = 1$ on $P(c,\eta)$. Hence $g(z) \neq 0$, for each $z \in P(c,\eta)$.

Now fix $z_2 \in D(c_2,\eta)$. Put $\psi_2(z_1, z_3, ..., z_{n+1}) = \psi_1(z_1, z_2, z_3, ..., z_{n+1})$, for $(z_1, z_3, ..., z_{n+1}) \in D(c_1,\eta) \times D(c_3,\eta) \times ... \times D(c_{n+1},\eta)$. ψ_2 is a holomorphic function in its domain. Since $(\psi_1)^s = 1$ on $P(c,\eta)$, ψ_2 is constant in the open polydisc $D(c_1,\eta) \times D(c_3,\eta) \times ... \times D(c_{n+1},\eta)$. It follows that $\psi_1(z_1, z_2, z_3, ..., z_{n+1}) = \varphi_1(z_2)$, for each $(z_1, z_2, z_3, ..., z_{n+1}) \in P(c,\eta)$, where $\varphi_1 : D(c_2,\eta) \to \mathbb{C}$, $(\varphi_1)^s = 1$. Consequently, $g^{\frac{1}{s}}(z_1, z_2, z_3, ..., z_{n+1}) = (\varphi_1(z_2)b_1(z_2)z_1 + \varphi_1(z_2)b_3(z_2)z_3 + \dots + \varphi_1(z_2)b_{n+1}(z_2)z_{n+1} + \varphi_1(z_2)b_{n+2}(z_2))$, for every $(z_1, z_2, z_3, ..., z_{n+1}) \in P(c,\eta)$.

Define $A_2 = \varphi a_2, ..., A_{n+1} = \varphi a_{n+1}$, $A_{n+2} = \varphi a_{n+2}$ on $D(c_1,\eta)$. Also $B_1 = \varphi_1 b_1, B_3 = \varphi_1 b_3, ..., B_{n+1} = \varphi_1 b_{n+1}$ and $B_{n+2} = \varphi_1 b_{n+2}$ on $D(c_2,\eta)$. Now let $Z_1^j \in D(c_2,\eta) \times ... \times D(c_{n+1},\eta)$, $1 \leq j \leq n$, $Z_1^j = (z_2^j, ..., z_{n+1}^j)$. Choose $(\xi_2^1, ..., \xi_{n+1}^1), ..., (\xi_2^n, ..., \xi_{n+1}^n) \in D(c_2,\eta) \times ... \times D(c_{n+1},\eta)$ such that the matrix

$$\begin{pmatrix} (z_2^1 - \xi_2^1) & (z_3^1 - \xi_3^1) & ... & (z_{n+1}^1 - \xi_{n+1}^1) \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ (z_2^n - \xi_2^n) & (z_3^n - \xi_3^n) & ... & (z_{n+1}^n - \xi_{n+1}^n) \end{pmatrix}$$

is invertible of type (n, n). Then we consider the 2 systems

$$(S) = \begin{cases} A_2(z_1)z_2^1 + A_3(z_1)z_3^1 + ... + A_{n+1}(z_1)z_{n+1}^1 + A_{n+2}(z_1) = g^{\frac{1}{s}}(z_1, Z_1^1), \\ . \\ . \\ . \\ A_2(z_1)z_2^n + A_3(z_1)z_3^n + ... + A_{n+1}(z_1)z_{n+1}^n + A_{n+2}(z_1) = g^{\frac{1}{s}}(z_1, Z_1^n), \end{cases}$$

for each $z_1 \in D(c_1, \eta)$ and

$$(S_1) = \begin{cases} A_2(z_1)\xi_2^1 + A_3(z_1)\xi_3^1 + ... + A_{n+1}(z_1)\xi_{n+1}^1 + A_{n+2}(z_1) = \\ g^{\frac{1}{s}}(z_1, \xi_2^1, ..., \xi_{n+1}^1), \\ . \\ . \\ . \\ A_2(z_1)\xi_2^n + A_3(z_1)\xi_3^n + ... + A_{n+1}(z_1)\xi_{n+1}^n + A_{n+2}(z_1) = \\ g^{\frac{1}{s}}(z_1, \xi_2^n, ..., \xi_{n+1}^n), \end{cases}$$

for each $z_1 \in D(c_1, \eta)$.
To calculate $A_2, ..., A_{n+1}$, we consider the difference between (S) and (S_1) denoted by (S_2).

$$(S_2) = \begin{cases} A_2(z_1)(z_2^1 - \xi_2^1) + A_3(z_1)(z_3^1 - \xi_3^1) + ... + A_{n+1}(z_1)(z_{n+1}^1 - \xi_{n+1}^1) = \\ g^{\frac{1}{s}}(z_1, Z_1^1) - g^{\frac{1}{s}}(z_1, \xi_2^1, ..., \xi_{n+1}^1) \\ . \\ . \\ . \\ A_2(z_1)(z_2^n - \xi_2^n) + A_3(z_1)(z_3^n - \xi_3^n) + ... + A_{n+1}(z_1)(z_{n+1}^n - \xi_{n+1}^n) = \\ g^{\frac{1}{s}}(z_1, Z_1^n) - g^{\frac{1}{s}}(z_1, \xi_2^n, ..., \xi_{n+1}^n) \end{cases}$$

for every $z_1 \in D(c_1, \eta)$. Thus, we calculate $A_2(z_1), A_3(z_1), ..., A_{n+1}(z_1)$ in function of $(g^{\frac{1}{s}}(z_1, Z_1^1) - g^{\frac{1}{s}}(z_1, \xi_2^1, ..., \xi_{n+1}^1)), ..., (g^{\frac{1}{s}}(z_1, Z_1^n) - g^{\frac{1}{s}}(z_1, \xi_2^n, ..., \xi_{n+1}^n))$, for every $z_1 \in D(c_1, \eta)$. It follows that $A_2, A_3, ..., A_{n+1}$ are holomorphic functions on $D(c_1, \eta)$. Now since $A_{n+2}(z_1) = -(A_2(z_1)z_2^1 + A_3(z_1)z_3^1 + ... + A_{n+1}(z_1)z_{n+1}^1) + g^{\frac{1}{s}}(z_1, Z_1^1)$, it follows that A_{n+2} is holomorphic on $D(c_1, \eta)$. Hence $g(z_1, z_2, z_3, ..., z_{n+1}) = (A_2(z_1)z_2 + A_3(z_1)z_3 + ... + A_{n+1}(z_1)z_{n+1} + A_{n+2}(z_1))^s$, for each $(z_1, z_2, z_3, ..., z_{n+1}) \in P(c, \eta)$.

Now by effectuate the same development, we prove that $B_1, B_3, ..., B_{n+1}, B_{n+2}$ are holomorphic functions on $D(c_2, \eta)$ and $g(z_1, z_2, z_3, ..., z_{n+1}) = (B_1(z_2)z_1 + B_3(z_2)z_3 + ... + B_{n+1}(z_2)z_{n+1} + B_{n+2}(z_2))^s$, for every $(z_1, z_2, z_3, ..., z_{n+1}) \in P(c, \eta)$. Consequently,

$$g(z_1, z_2, z_3, ..., z_{n+1}) = (A_2(z_1)z_2 + A_3(z_1)z_3 + ... + A_{n+1}(z_1)z_{n+1} + A_{n+2}(z_1))^s$$

$$= (B_1(z_2)z_1 + B_3(z_2)z_3 + ... + B_{n+1}(z_2)z_{n+1} + B_{n+2}(z_2))^s,$$

for each $(z_1, z_2, z_3, ..., z_{n+1}) \in P(c, \eta)$. Then there exists $\lambda : P(c, \eta) \to \mathbb{C}$ such that $(\lambda(z))^s = 1$ and $(A_2(z_1)z_2 + A_3(z_1)z_3 + ... + A_{n+1}(z_1)z_{n+1} + A_{n+2}(z_1)) = \lambda(z)(B_1(z_2)z_1 + B_3(z_2)z_3 + ... + B_{n+1}(z_2)z_{n+1} + B_{n+2}(z_2))$, for any $z = (z_1, z_2, z_3, ..., z_{n+1}) \in P(c, \eta)$. Since $g(z) \neq 0$, for all $z \in P(c, \eta)$, then $\lambda(z) \neq 0$, for each $z \in P(c, \eta)$. Therefore λ is a holomorphic function on $P(c, \eta)$. Since $\lambda^s = 1$ on $P(c, \eta)$, λ is constant on $P(c, \eta)$ and $\lambda \neq 0$. Thus

$$A_2(z_1)z_2 + A_3(z_1)z_3 + ... + A_{n+1}(z_1)z_{n+1} + A_{n+2}(z_1) =$$

$$\lambda(B_1(z_2)z_1 + B_3(z_2)z_3 + ... + B_{n+1}(z_2)z_{n+1} + B_{n+2}(z_2)),$$

for each $z = (z_1, z_2, z_3, ..., z_{n+1}) \in P(c, \eta)$. The derivative relative to z_j (for $j \in \{3, ..., n+1\}$) implies that $A_j(z_1) = \lambda B_j(z_2)$, for each $z_1 \in D(c_1, \eta)$, $z_2 \in D(c_2, \eta)$. Since $\lambda \neq 0$, A_j and B_j are constant functions respectively on $D(c_1, \eta)$ and $D(c_2, \eta)$, for all $j \in \{3, ..., n+1\}$. The derivative relative to z_2 implies that $A_2(z_1) = \lambda B_1'(z_2)z_1 + \lambda B_{n+2}'(z_2)$, for all $z_1 \in D(c_1, \eta)$ and $z_2 \in D(c_2, \eta)$. Hence $A_2'(z_1) = \lambda B_1'(z_2)$, for each $(z_1, z_2) \in D(c_1, \eta) \times D(c_2, \eta)$. Since $\lambda \neq 0$, A_2' and B_1' are constant functions respectively on $D(c_1, \eta)$ and $D(c_2, \eta)$. Thus the functions A_2 and B_1 are affine functions on their above domains. Now we derive the equality

$$A_2(z_1) = \lambda B_1'(z_2)z_1 + \lambda B_{n+2}'(z_2)$$

relative to the variable $z_2 \in D(c_2, \eta)$. Then $0 = \lambda B_{n+2}''(z_2)$, for all $z_2 \in D(c_2, \eta)$. Thus the function B_{n+2} is affine on $D(c_2, \eta)$.

Finally consider the equality

$$A_2(z_1)z_2 + A_3(z_1)z_3 + ... + A_{n+1}(z_1)z_{n+1} + A_{n+2}(z_1) =$$

$$\lambda(B_1(z_2)z_1 + B_3(z_2)z_3 + ... + B_{n+1}(z_2)z_{n+1} + B_{n+2}(z_2))$$

for all $z = (z_1, z_2, z_3, ..., z_{n+1}) \in P(c, \eta)$. The derivative relative to z_1 implies that $A_2'(z_1)z_2 + A_{n+2}'(z_1) = \lambda B_1(z_2)$, for all $z_1 \in D(c_1, \eta)$, for all $z_2 \in D(c_2, \eta)$. Since A_2' is constant on $D(c_1, \eta)$, $A_{n+2}''(z_1) = 0$, for each $z_1 \in D(c_1, \eta)$. Consequently, A_{n+2} is affine on $D(c_1, \eta)$. Now we have $A_3, ..., A_{n+1}$ are constant functions on $D(c_1, \eta)$. Thus A_2 and A_{n+2} are affine functions on $D(c_1, \eta)$. Then $A_2(z_1) = \lambda_2 z_1 + \mu_2$, $A_{n+2}(z_1) = \lambda_{n+2} z_1 + \mu_{n+2}$, $A_3(z_1) = \mu_3, ..., A_{n+1}(z_1) = \mu_{n+1}$, for all $z_1 \in D(c_1, \eta)$, where $\lambda_2, \mu_2, \lambda_{n+2}, \mu_{n+2}, \mu_3, ..., \mu_{n+1} \in \mathbb{C}$.
So $g(z_1, z_2, z_3, ..., z_{n+1}) = [(\lambda_2 z_1 + \mu_2)z_2 + \mu_3 z_3 + ... + \mu_{n+1} z_{n+1} + \lambda_{n+2} z_1 + \mu_{n+2}]^s$, for each $(z_1, z_2, z_3, ..., z_{n+1}) \in P(c, \eta)$. Define

$$f(z) = [(\lambda_2 z_1 + \mu_2)z_2 + \mu_3 z_3 + ... + \mu_{n+1} z_{n+1} + \lambda_{n+2} z_1 + \mu_{n+2}]^s$$

for $z = (z_1, z_2, z_3, ..., z_{n+1}) \in \mathbb{C}^{n+1}$. Then f is a holomorphic function on $\mathbb{C}^{n+1}$. And so $g = f$ on $P(c, \eta) = D(c_1, \eta) \times ... \times D(c_{n+1}, \eta)$. Moreover, $g = f$ on $\mathbb{C}^{n+1}$.

Now we prove that $\lambda_2 = 0$. Assume that $\lambda_2 \neq 0$. Then $K(z_1, z_2) = g(z_1, z_2, 0, ..., 0) = (\lambda_2 z_1 z_2 + \lambda_{n+2} z_1 + \mu_2 z_2 + \mu_{n+2})^s$, for all $(z_1, z_2) \in \mathbb{C}^2$, K is a holomorphic function on $\mathbb{C}^2$ and K satisfy $|K|$ is convex on $\mathbb{C}^2$. This is a contradiction. Consequently, $\lambda_2 = 0$. The proof is now finished. □

Applications. We can use theorem 3.7 and theorem 3.1 for the resolution of several holomorphic partial differential equations.

Example 3.8. (A) Find all the holomorphic functions $g : \mathbb{C}^n \to \mathbb{C}$, $(n \geq 2)$ such that u is convex on $\mathbb{C}^n$, where $u(z) = |a\frac{\partial^2 g}{\partial z_1^2}(z) + b\frac{\partial g}{\partial z_1}(z) + cg(z)|$, $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $a, b \in \mathbb{C}\backslash\{0\}$ and $c \in \mathbb{C}$.

(B) Find all the analytic functions $g : \mathbb{C}^n \to \mathbb{C}$, $(n \geq 2)$ such that v is convex on $\mathbb{C}^n$, $v(z) = e^{|z_1 \frac{\partial g}{\partial z_2}(z) + z_2 \frac{\partial g}{\partial z_1}(z) + g(z)|}$, for $z = (z_1, ..., z_n) \in \mathbb{C}^n$.

(C) Let $\varphi : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function, $n \geq 1$ and $\delta > 0$.

We say that $|\varphi|$ is $\delta-$ convex on $\mathbb{C}^n$ if $|\varphi(tz + (1-t)\xi)| \leq t|\varphi(z)| + (1-t)|\varphi(\xi)| + \delta$, for all $z, \xi \in \mathbb{C}^n$, for all $t \in [0, 1]$. Now we can use the above contribution for the study of the family

$$E = \{|g| \ / \ g : \mathbb{C}^n \to \mathbb{C} \text{ be holomorphic and } \ |g| \text{ is } \delta - \text{ convex on } \mathbb{C}^n\}$$

where $\delta > 0$ and $n \geq 1$.

Various authors have studied the relation between the pluripolarity of the graph of a continuous function and the analyticity property, we can see [22], [7], [8] and [9]. On the other hand we can study the problem

$$\begin{cases} g : \mathbb{C}^n \backslash F \to \mathbb{C} \text{ be holomorphic and} \\ |g| \text{ is convex on } \mathbb{C}^n \backslash F \end{cases}$$

where F is a closed subset of $\mathbb{C}^n$, $n \geq 1$ and $H^{2n-1}(F) = 0$, H^{2n-1} is the $(2n-1)$ dimensional Hausdorff measure [5]. The paper described by Cegrell [5] deals to this subject (we can see also El Mir [10]).

Corollary 3.9. *Let $g : \mathbb{C}^n \to \mathbb{C}$ be an analytic function, $n \geq 1$, $a, b \in \mathbb{C}$, $(a \neq b)$. We have the technical assertions*

(A) *Assume that $|g + a|$ and $|g + b|$ are convex functions on $\mathbb{C}^n$. Then g is affine on $\mathbb{C}^n$.*

(B) *Assume that $|g^2 + a|$ and $|g^2 + b|$ are convex functions on $\mathbb{C}^n$. Then g is constant on $\mathbb{C}^n$.*

(C) *Assume that there exist $a_1, ..., a_N \in \mathbb{C}^n$, where $N \in \mathbb{N}\backslash\{0\}$ such that the function $(|g + a_1|^2 + \ldots + |g + a_N|^2)$ is convex on $\mathbb{C}^n$. Then $|g + \frac{a_1 + \ldots + a_N}{N}|$ is convex on $\mathbb{C}^n$.*

Note that if $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ are two holomorphic nonconstant functions, with $|g_1|$, $|g_2|$, $|g_1 - g_2|$, $|g_1 + g_2|$ are convex functions on $\mathbb{C}^n$ $(n \geq 1)$, we can state some properties concerning g_1 and g_2.

Corollary 3.10. *Let $(A_1, A_2), (B_1, B_2) \in \mathbb{C}^2$ such that $\{(A_1, A_2), (B_1, B_2)\}$ is a free family on $\mathbb{C}^2$. Given $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be 2 analytic functions, $n \geq 1$. Define*

$$u(z, w) = |A_1 w - g_1(z)|^2 + |A_2 w - g_2(z)|^2,$$
$$v(z, w) = |B_1 w - g_1(z)|^2 + |B_2 w - g_2(z)|^2$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The following assertions are equivalent:

(A) *u and v are convex functions on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *g_1 and g_2 are affine functions on $\mathbb{C}^n$.*

Proof. (A) implies (B). Since u is convex on $\mathbb{C}^n \times \mathbb{C}$, by [3] we have

$$\begin{cases} g_1(z) = A_1(< z/a_1 > + a_2) + \overline{A_2}\varphi(z) \\ g_2(z) = A_2(< z/a_1 > + a_2) - \overline{A_1}\varphi(z) \end{cases}$$

for all $z \in \mathbb{C}^n$, where $a_1 \in \mathbb{C}^n$, $a_2 \in \mathbb{C}$, $\varphi : \mathbb{C}^n \to \mathbb{C}$, φ is analytic and $|\varphi|$ is convex on $\mathbb{C}^n$. Also v is convex on $\mathbb{C}^n \times \mathbb{C}$, then

$$\begin{cases} g_1(z) = B_1(< z/b_1 > + b_2) + \overline{B_2}\psi(z) \\ g_2(z) = B_2(< z/b_1 > + b_2) - \overline{B_1}\psi(z) \end{cases}$$

for all $z \in \mathbb{C}^n$, with $b_1 \in \mathbb{C}^n$, $b_2 \in \mathbb{C}$, $\psi : \mathbb{C}^n \to \mathbb{C}$, ψ is analytic and $|\psi|$ is convex on $\mathbb{C}^n$. We would like to prove that φ and ψ are affine functions on $\mathbb{C}^n$. We have

$$\begin{cases} \overline{A_2}\varphi(z) - \overline{B_2}\psi(z) = < z/\lambda > + \mu \\ -\overline{A_1}\varphi(z) + \overline{B_1}\psi(z) = < z/\lambda_1 > + \mu_1 \end{cases}$$

for all $z \in \mathbb{C}^n$, with $\lambda, \lambda_1 \in \mathbb{C}^n$ and $\mu, \mu_1 \in \mathbb{C}$. Since the determinant $\det((\overline{A_2}, -\overline{A_1}), (-\overline{B_2}, \overline{B_1})) \neq 0$, then we calculate $\varphi(z)$ and $\psi(z)$ in function of $(< z/\lambda > + \mu)$ and $(< z/\lambda_1 > + \mu_1)$, for all fixed $z \in \mathbb{C}^n$. Therefore φ and ψ are affine functions on $\mathbb{C}^n$. Consequently, g_1 and g_2 are affine functions on $\mathbb{C}^n$.

(B) implies (A). This case is obvious. □

4 Some applications and algebraic methods

Definition 4.1. Let $u : D \to \mathbb{R}$ be a function of class C^2, where D is a domain of $\mathbb{C}^n$, $n \geq 1$. Let $a \in D$. Then u is called strictly psh at a if $\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}(a)\alpha_j\overline{\alpha_k} > 0$, for every $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n \backslash \{0\}$, and u is called strictly psh on D if u is strictly psh at all points $z \in D$.

Lemma 4.2. *Let D be a domain of $\mathbb{C}^n$, $n, N \geq 1$. Consider $(2N)$ holomorphic functions $f_1, ..., f_N, g_1, ..., g_N : D \to \mathbb{C}$. Define $u = (|f_1|^2 + \ldots + |f_N|^2)$.*

(A) *Assume that u is strictly psh on D. Then $n \leq N$.*

(B) *Assume that $N < n$. Then u is not strictly psh at any point of D.*

(C) *$(|f_1 - \overline{g_1}|^2 + ... + |f_N - \overline{g_N}|^2)$ is strictly psh on D if and only if $(|f_1|^2 + |g_1|^2 + ... + |f_N|^2 + |g_N|^2)$ is strictly psh on D. (We have the same equivalence for the case strictly sh). Indeed, if $(|f_1 - \overline{g_1}|^2 + ... + |f_N - \overline{g_N}|^2)$ is strictly psh on D, then $2N \geq n$.*

Proof. Note that u is a function of class C^∞ on D. The hermitian Levi form of u is

$$L(u)(z)(\alpha) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k} = |\sum_{j=1}^{n} \frac{\partial f_1}{\partial z_j}(z)\alpha_j|^2 + ... + |\sum_{j=1}^{n} \frac{\partial f_N}{\partial z_j}(z)\alpha_j|^2$$

for all $z = (z_1, ..., z_n) \in D$, for each $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$.

(A). Fix $z \in D$. We have the condition $L(u)(z)(\alpha) = 0$ and $\alpha \in \mathbb{C}^n$ implies that $\alpha = 0$. Hence,

$$\begin{cases} \alpha_1 \frac{\partial f_1}{\partial z_1}(z) + ... + \alpha_n \frac{\partial f_1}{\partial z_n}(z) = 0 \\ . \\ . \\ . \\ \alpha_1 \frac{\partial f_N}{\partial z_1}(z) + ... + \alpha_n \frac{\partial f_N}{\partial z_n}(z) = 0 \end{cases}$$

implies that $\alpha_1 = ... = \alpha_n = 0$. Then if $\alpha_1(\frac{\partial f_1}{\partial z_1}(z), ..., \frac{\partial f_N}{\partial z_1}(z)) + \ldots + \alpha_n(\frac{\partial f_1}{\partial z_n}(z), ..., \frac{\partial f_N}{\partial z_n}(z)) = (0, ..., 0) \in \mathbb{C}^N$ and $(\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$, then $\alpha_1 = ... = \alpha_n = 0$. Thus the subset of n vectors $\{(\frac{\partial f_1}{\partial z_1}(z), ..., \frac{\partial f_N}{\partial z_1}(z)), ..., (\frac{\partial f_1}{\partial z_n}(z), ..., \frac{\partial f_N}{\partial z_n}(z))\}$ is a free family on $\mathbb{C}^N$. Since $\mathbb{C}^n$ is a complex vector space of dimension N, then $n \leq N$.

(B). Fix $z = (z_1, ..., z_n) \in D$ and assume that the hermitian Levi form of u satisfy $L(u)(z)(\alpha) > 0$, for all $\alpha \in \mathbb{C}^n \backslash \{0\}$. Therefore the condition $L(u)(z)(\alpha) = 0$ implies that $\alpha = 0$. But $L(u)(z)(\alpha) = |\sum_{j=1}^{n} \frac{\partial f_1}{\partial z_j}(z)\alpha_j|^2 + ... + |\sum_{j=1}^{n} \frac{\partial f_N}{\partial z_j}(z)\alpha_j|^2$, for $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$. Now $L(u)(z)(\alpha) = 0$ implies that $\sum_{j=1}^{n} \frac{\partial f_1}{\partial z_j}(z)\alpha_j = 0, ..., \sum_{j=1}^{n} \frac{\partial f_N}{\partial z_j}(z)\alpha_j = 0$. Thus

$$\begin{cases} \alpha_1 \frac{\partial f_1}{\partial z_1}(z) + ... + \alpha_n \frac{\partial f_1}{\partial z_n}(z) = 0 \\ . \\ . \\ . \\ \alpha_1 \frac{\partial f_N}{\partial z_1}(z) + ... + \alpha_n \frac{\partial f_N}{\partial z_n}(z) = 0 \end{cases}$$

and so $\alpha_1(\frac{\partial f_1}{\partial z_1}(z), ..., \frac{\partial f_N}{\partial z_1}(z)) + ... + \alpha_n(\frac{\partial f_1}{\partial z_n}(z), ..., \frac{\partial f_N}{\partial z_n}(z)) = 0 \in \mathbb{C}^N$ implies that $\alpha_1 = ... = \alpha_n = 0$. Then the subset of n vectors $\{(\frac{\partial f_1}{\partial z_1}(z), ..., \frac{\partial f_N}{\partial z_1}(z)), ..., (\frac{\partial f_1}{\partial z_n}(z), ..., \frac{\partial f_N}{\partial z_n}(z))\}$ is a free family of the vector space $\mathbb{C}^N$, $\mathbb{C}^N$ is a complex vector space of dimension N and $N < n$. This is a contradiction. Consequently, for all $z \in D$, there exists $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n \backslash \{0\}$ such that $L(u)(z)(\alpha) = 0$. Then u is not strictly psh at all points of D.

(C). We have $(|f_1 - \overline{g_1}|^2 + ... + |f_N - \overline{g_N}|^2)$ is strictly psh on D if and only if $v = (|f_1|^2 + |g_1|^2 + ... + |f_N|^2 + |g_N|^2)$ is strictly psh on D, because $(|f_1 - \overline{g_1}|^2 + ... + |f_N - \overline{g_N}|^2) = (h + v)$ on D, where $h : D \to \mathbb{R}$ is a prh function. By (A), we have $2N \geq n$. □

Proposition 4.3. *Let $g_1, ..., g_N : D \to \mathbb{C}$ be N analytic functions, $n, N \geq 1$ and D is a domain of $\mathbb{C}^n$. The following conditions are equivalent*

(A) $u = (e^{|g_1|^2} + ... + e^{|g_N|^2})$ *is strictly psh on D and $n \geq N$;*

(B) *For all $z \in D$, the subset $\{(\frac{\partial g_1}{\partial z_1}(z), ..., \frac{\partial g_N}{\partial z_1}(z)), ..., (\frac{\partial g_1}{\partial z_n}(z), ..., \frac{\partial g_N}{\partial z_n}(z))\}$ is a generating family on $\mathbb{C}^N$.*

Let $m \geq 2$. Now recall that for all harmonic functions $h : G \to \mathbb{R}$, we have h is not convex on all open balls subset of G, where G is an open of $\mathbb{R}^m$, if h is not affine on G.

Theorem 4.4. *(A) Let $g : \mathbb{C} \to \mathbb{C}$ be an analytic function. Then there exists an open disc $D(z_0, r)$, $(z_0 \in \mathbb{C}, r > 0)$ such that $|g|^2$ is convex on $D(z_0, r)$.*

(B) For any $h : \mathbb{C} \to \mathbb{R}$ be a harmonic function, there exists an open disc $D(a, R)$, $a \in \mathbb{C}$ and $R > 0$ such that u is psh on the convex not bounded domain $G = \{(z, w) \in \mathbb{C}^2/(w - \overline{z}) \in D(a, R)\}$, but u_1 is not psh on all not empty open ball subset of G if h is not affine on $\mathbb{C}$, where $u(z, w) = e^{h(w - \overline{z})}$ and $u_1(z, w) = h(w - \overline{z})$, for $(z, w) \in \mathbb{C}^2$.

This theorem have many applications in the case of the characterization of holomorphic functions by plurisubharmonic functions (which is a fundamental subject in pluripotential theory).

Proof. (A). Assume that $|g|^2$ is not convex on any not empty open disc subset of $\mathbb{C}$. Therefore there exists $\zeta_1 \in \mathbb{C}$, such that $|g''(\zeta_1)g(\zeta_1)| > |g'(\zeta_1)|^2$. Now since $|g''g| \geq |g'|^2$ on $\mathbb{C}$, then $g''(z)g(z) = \gamma(g'(z))^2$, for each $z \in \mathbb{C}$, where $\gamma \in \mathbb{C}$, $|\gamma| \geq 1$. By [2, Theorem 21], we have $\gamma \in \{\frac{s-1}{s}, 1 \ / \ s \in \mathbb{N}\backslash\{0\}\}$. Therefore $\gamma = 1$. The condition $|g''(\zeta_1)g(\zeta_1)| > |g'(\zeta_1)|^2$ implies that $|\gamma| > 1$, a contradiction. Consequently, there exists an open disc $D(z_0, r)$ where $z_0 \in \mathbb{C}$ and $r > 0$, such that $|g|^2$ is convex on $D(z_0, r)$.

(B). Let $k : \mathbb{C} \to \mathbb{C}$, k is analytic and $\mathrm{Re}(k) = h$. Then e^k is analytic on $\mathbb{C}$ and $e^h = |e^k|$. Thus there exists an open disc $D(z_0, r)$ $(z_0 \in \mathbb{C}, r > 0)$, such that $|e^k|$ is convex on $D(z_0, r)$. It follows that u is psh on G. Assume that u_1 is psh on an open ball $B((z_1, w_1), R_1) \subset G$, $(z_1, w_1) \in \mathbb{C}^2$, $R_1 > 0$. Define $a = w_1 - \overline{z_1}$. Then $a \in D(z_0, r)$. Now let $(z_2, w_2) \in \mathbb{C}^2$ such that $w_2 - \overline{z_2} = a$. Thus u_1 is psh on $B((z_2, w_2), R_1)$. Consequently, u_1 is psh on $G_1 = \{(z, w) \in \mathbb{C}^2 \ / \ |w - \overline{z} - a| < R_1\} = \{(z, w) \in \mathbb{C}^2 \ / \ (w - \overline{z}) \in D(a, R_1)\}$. It follows that h is convex on $D(a, R_1)$. But h is not an affine function on $\mathbb{C}$, we get a contradiction. □

Remark 4.5. Let $D = D(0, 1)$ and $g(z) = z^2 + 4$, for $z \in D(0, 1)$. g is holomorphic on D. But $|g|^2$ is not convex on each not empty open disc subset of D. It follows that in all non- empty bounded convex domains $D_1 \subset \mathbb{C}$, there exists an holomorphic function $g_1 : D_1 \to \mathbb{C}$, $|g_1|^2$ is not convex in all non-empty open discs of D_1. In fact let $R > 0$ such that $D_1 \subset D(0, R)$ and we can consider in this case $g_1(z) = g(\frac{z}{R})$ for $z \in D(0, R)$.

Theorem 4.6. *Let $A_1, B_1, ..., A_N, B_N \in \mathbb{C}$, $n, s, N \in \mathbb{N}\backslash\{0\}$. Put $v = (v_1 + ... + v_N)$, $\psi = (\psi_1 + ... + \psi_N)$, where $f_1, g_1, ..., f_N, g_N : \mathbb{C}^n \to \mathbb{C}$ are $2N$ analytic functions and $u_j(z,w) = |A_jw - f_j(z)|^2 + |B_jw - g_j(z)|^2$, $v_j(z,w) = |A_jw - \frac{\partial^s f_j}{\partial z_1^s}(z)|^2 + |B_jw - \frac{\partial^s g_j}{\partial z_1^s}(z)|^2$, $\psi_j(z,w) = |A_jw - \overline{\frac{\partial^s f_j}{\partial z_1^s}}(z)|^2 + |B_jw - \overline{\frac{\partial^s g_j}{\partial z_1^s}}(z)|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, $j \in \{1, ..., N\}$. The following three conditions are equivalent:*

(A) *$u_1, ..., u_N$ are convex functions on $\mathbb{C}^n \times \mathbb{C}$ and v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *For $n \leq N$ and for all $j \in \{1, ..., N\}$, we have the holomorphic representation*

$$\begin{cases} f_j(z) = A_j(< z/a_j > +b_j) + \overline{B_j}\varphi_j(z) \\ g_j(z) = B_j(< z/a_j > +b_j) - \overline{A_j}\varphi_j(z) \end{cases}$$

for all $z \in \mathbb{C}^n$, where $a_j \in \mathbb{C}^n$, $b_j \in \mathbb{C}$, $\varphi_j : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function, $|\varphi_j|$ is convex on $\mathbb{C}^n$, such that the set of n vectors

$$\{(\frac{\partial^{s+1}\varphi_1}{\partial z_1^{s+1}}(z), ..., \frac{\partial^{s+1}\varphi_N}{\partial z_1^{s+1}}(z)), (\frac{\partial^{s+1}\varphi_1}{\partial z_1^s \partial z_2}(z), ..., \frac{\partial^{s+1}\varphi_N}{\partial z_1^s \partial z_2}(z)), ..., (\frac{\partial^{s+1}\varphi_1}{\partial z_1^s \partial z_n}(z), ..., \frac{\partial^{s+1}\varphi_N}{\partial z_1^s \partial z_n}(z))\}$$

is a free family of the complex vector space $\mathbb{C}^N$, for all $z \in \mathbb{C}^n$;

(C) *$u_1, ..., u_N$ are convex functions on $\mathbb{C}^n \times \mathbb{C}$ and ψ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.*

Note that we have another generalization of the above theorem for each holomorphic partial differential equation having constant coefficients on $\mathbb{C}^n$, $n \geq 1$.

Question 4.7. Let $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4 \in \mathbb{C}\backslash\{0\}$ and $f_1, g_1, f_2, g_2, f_3, g_3, f_4, g_4 : \mathbb{C}^n \to \mathbb{C}$ be 8 holomorphic functions, $n \geq 1$. Define $u_j(z,w) = |A_jw - f_j(z)|^2 + |B_jw - g_j(z)|^2$, $v_j(z,w) = |A_jw - \overline{f_j}(z)|^2 + |B_jw - \overline{g_j}(z)|^2$ for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ and $j \in \{1,2,3,4\}$. Let $u = (u_1 + u_2 + u_3 + u_4)$, $v = (v_1 + v_2 + v_3 + v_4)$ and $\varphi = u + v$.

(A) Find exactly all the holomorphic functions $f_1, g_1, f_2, g_2, f_3, g_3, f_4, g_4 : \mathbb{C}^n \to \mathbb{C}$ such that

$$\begin{cases} u_1, u_2, u_3 \text{ and } u_4 \text{ are convex functions on } \mathbb{C}^n \times \mathbb{C}, \\ u \text{ is not strictly psh on all open balls of } \mathbb{C}^n \times \mathbb{C}, \\ \varphi \text{ is strictly psh on } \mathbb{C}^n \times \mathbb{C}. \end{cases}$$

What can we say of n?

(B) Find all the holomorphic functions $f_1, g_1, f_2, g_2, f_3, g_3, f_4, g_4 : \mathbb{C}^n \to \mathbb{C}$ such that

$$\begin{cases} u_1 \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and not strictly psh on } B(a_1, R_1) \times D(b_1, r_1), \\ u_2 \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and not strictly psh on } B(a_2, R_2) \times D(b_2, r_2), \\ u_3 \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and not strictly psh on } B(a_3, R_3) \times D(b_3, r_3), \\ u_4 \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and not strictly psh on } B(a_4, R_4) \times D(b_4, r_4), \\ u \text{ is } (n+1)- \text{ strictly sh on } \mathbb{C}^n \times \mathbb{C} \text{ but not strictly psh on all open balls} \\ \text{of } \mathbb{C}^n \times \mathbb{C}, \text{ and} \\ \varphi \text{ is strictly psh on } \mathbb{C}^n \times \mathbb{C}. \end{cases}$$

$a_1, a_2, a_3, a_4 \in \mathbb{C}^n, b_1, b_2, b_3, b_4 \in \mathbb{C}$, $R_1, R_2, R_3, R_4, r_1, r_2, r_3, r_4 \in \mathbb{R}_+\backslash\{0\}$.

Lemma 4.8. *Let $A_1, ..., A_N \in \mathbb{C}$, $f_1, ..., f_N : \mathbb{C}^n \to \mathbb{C}$ be N analytic functions, $n, N \geq 1$. Define $u_j(z,w) = |A_jw - f_j(z)|^2$, $v_j(z,w) = |A_jw - \overline{f_j}(z)|^2$, $u = (u_1 + ... + u_N)$, $v = (v_1 + ... + v_N)$ and $\varphi = (u + v)$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ and $j \in \{1, ..., N\}$. The following conditions are equivalent:*

(A) *φ is not strictly psh on all not empty open balls of $\mathbb{C}^n \times \mathbb{C}$;*

(B) *v is not strictly psh on any nonempty Euclidean open ball of $\mathbb{C}^n \times \mathbb{C}$.*

Proof. We have φ and v are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. The Levi Hermitian form of φ is $L(\varphi)(z,w)(\alpha,\beta) = \sum_{j=1}^{N} |A_j\beta - \sum_{k=1}^{n} \frac{\partial f_j}{\partial z_k}(z)\alpha_k|^2 + \sum_{j=1}^{N}(|A_j\beta|^2 + |\sum_{k=1}^{n} \frac{\partial f_j}{\partial z_k}(z)\alpha_k|^2)$, for $(z,w) = (z_1, ..., z_n, w)$, $(\alpha,\beta) = (\alpha_1, ..., \alpha_n, \beta) \in \mathbb{C}^n \times \mathbb{C}$. Similarly, $L(v)(z,w)(\alpha,\beta) = \sum_{j=1}^{N}(|A_j\beta|^2 + |\sum_{k=1}^{n} \frac{\partial f_j}{\partial z_k}(z)\alpha_k|^2)$. Note that $L(\varphi)(z,w)(\alpha,\beta)$ and $L(v)(z,w)(\alpha,\beta)$ are functions independent of w. Observe that we have the inequalities

$$0 \leq L(v)(z,w)(\alpha,\beta) \leq L(\varphi)(z,w)(\alpha,\beta) \leq 3L(v)(z,w)(\alpha,\beta)$$

for all $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, for each $(\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C}$.

(A) implies (B). Let $z \in \mathbb{C}^n$ and $R > 0$. There exist $z^0 \in B(z,R)$, $(\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C}$, such that $L(\varphi)(z,w)(\alpha,\beta) = 0$, for all $w \in \mathbb{C}$. Then $L(v)(z,w)(\alpha,\beta) = 0$. Thus, for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, for every $R > 0$, there exist $(z^0,w) \in \mathbb{C}^n \times \mathbb{C}$, $(\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C}$, such that $L(v)(z,w)(\alpha,\beta) = 0$. Hence, v is not strictly psh in all Euclidean open balls of $\mathbb{C}^n \times \mathbb{C}$.

(B) implies (A). This is obvious by the above proof. □

As an application, we have the following.

Theorem 4.9. *Let $A_1, B_1, A_2, B_2, A_3, B_3, A_4, B_4 \in \mathbb{C}\backslash\{0\}$. Let $f_1, g_1, f_2, g_2, f_3, g_3, f_4, g_4 : \mathbb{C}^n \to \mathbb{C}$ be 8 analytic functions. Define*

$$\begin{cases} u_j(z,w) = |A_j w - f_j(z)|^2 + |B_j w - g_j(z)|^2, \\ v_j(z,w) = |A_j w - \overline{f_j}(z)|^2 + |B_j w - \overline{g_j}(z)|^2 \end{cases}$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ and $j \in \{1,2,3,4\}$. $u = (u_1 + u_2 + u_3 + u_4)$, $v = (v_1 + v_2 + v_3 + v_4)$, $\varphi = (u+v)$. The following conditions are equivalent:

(A) *u_1, u_2, u_3 and u_4 are convex functions on $\mathbb{C}^n \times \mathbb{C}$, u is not strictly psh in all open balls of $\mathbb{C}^n \times \mathbb{C}$ and φ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *$1 \leq n \leq 8$, u is not strictly psh at all points of $\mathbb{C}^n \times \mathbb{C}$ and for all $j \in \{1,2,3,4\}$, we have the holomorphic representation*

$$\begin{cases} f_j(z) = A_j(< z/\lambda_j > + \mu_j) + \overline{B_j}\varphi_j(z) \\ g_j(z) = B_j(< z/\lambda_j > + \mu_j) - \overline{A_j}\varphi_j(z) \end{cases}$$

for all $z \in \mathbb{C}^n$, where $\lambda_j \in \mathbb{C}^n$, $\mu_j \in \mathbb{C}$, $\varphi_j : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function, $|\varphi_j|$ is convex on $\mathbb{C}^n$, $\lambda_j = (\lambda_{j1}, ..., \lambda_{jn})$, with the following statements.

(i) *For $n = 8$. We have $\{(\lambda_1, \lambda_2, \lambda_3, \lambda_4, (\overline{\frac{\partial\varphi_1}{\partial z_1}}(z), ..., \overline{\frac{\partial\varphi_1}{\partial z_8}}(z)), (\overline{\frac{\partial\varphi_2}{\partial z_1}}(z), ..., \overline{\frac{\partial\varphi_2}{\partial z_8}}(z)), (\overline{\frac{\partial\varphi_3}{\partial z_1}}(z), ..., \overline{\frac{\partial\varphi_3}{\partial z_8}}(z)), (\overline{\frac{\partial\varphi_4}{\partial z_1}}(z), ..., \overline{\frac{\partial\varphi_4}{\partial z_8}}(z)))\}$ is a basis of the complex vector space $\mathbb{C}^8$, for all $z \in \mathbb{C}^8$.*

(ii) *For* $n=7$. *We have* $\{(\overline{\lambda_{11}},\overline{\lambda_{21}},\overline{\lambda_{31}},\overline{\lambda_{41}},\frac{\partial\varphi_1}{\partial z_1}(z),\frac{\partial\varphi_2}{\partial z_1}(z),\frac{\partial\varphi_3}{\partial z_1}(z),\frac{\partial\varphi_4}{\partial z_1}(z)),...,$ $(\overline{\lambda_{17}},\overline{\lambda_{27}},\overline{\lambda_{37}},\overline{\lambda_{47}},\frac{\partial\varphi_1}{\partial z_7}(z),\frac{\partial\varphi_2}{\partial z_7}(z),\frac{\partial\varphi_3}{\partial z_7}(z),\frac{\partial\varphi_4}{\partial z_7}(z))\}$ *is a free family on* $\mathbb{C}^8$, *for all* $z\in\mathbb{C}^7$ *and* $\{(\overline{\lambda_{11}},\overline{\lambda_{21}},\overline{\lambda_{31}},\overline{\lambda_{41}},\frac{\partial\varphi_1}{\partial z_1}(z),\frac{\partial\varphi_2}{\partial z_1}(z),\frac{\partial\varphi_3}{\partial z_1}(z),\frac{\partial\varphi_4}{\partial z_1}(z)),...,$ $(\overline{\lambda_{17}},\overline{\lambda_{27}},\overline{\lambda_{37}},\overline{\lambda_{47}},\frac{\partial\varphi_1}{\partial z_7}(z),\frac{\partial\varphi_2}{\partial z_7}(z),\frac{\partial\varphi_3}{\partial z_7}(z),\frac{\partial\varphi_4}{\partial z_7}(z)),(1,1,1,1,0,0,0,0)\}$ *is not a free family on* $\mathbb{C}^8$, *for all* $z\in\mathbb{C}^7$.

(iii) *For* $n=6$. *We have* $\{(\overline{\lambda_{11}},\overline{\lambda_{21}},\overline{\lambda_{31}},\overline{\lambda_{41}},\frac{\partial\varphi_1}{\partial z_1}(z),\frac{\partial\varphi_2}{\partial z_1}(z),\frac{\partial\varphi_3}{\partial z_1}(z),\frac{\partial\varphi_4}{\partial z_1}(z)),...,$ $(\overline{\lambda_{16}},\overline{\lambda_{26}},\overline{\lambda_{36}},\overline{\lambda_{46}},\frac{\partial\varphi_1}{\partial z_6}(z),\frac{\partial\varphi_2}{\partial z_6}(z),\frac{\partial\varphi_3}{\partial z_6}(z),\frac{\partial\varphi_4}{\partial z_6}(z))\}$ *is a free family on* $\mathbb{C}^8$, *for all* $z\in\mathbb{C}^6$ *and* $\{(\overline{\lambda_{11}},\overline{\lambda_{21}},\overline{\lambda_{31}},\overline{\lambda_{41}},\frac{\partial\varphi_1}{\partial z_1}(z),\frac{\partial\varphi_2}{\partial z_1}(z),\frac{\partial\varphi_3}{\partial z_1}(z),\frac{\partial\varphi_4}{\partial z_1}(z)),...,$ $(\overline{\lambda_{16}},\overline{\lambda_{26}},\overline{\lambda_{36}},\overline{\lambda_{46}},\frac{\partial\varphi_1}{\partial z_6}(z),\frac{\partial\varphi_2}{\partial z_6}(z),\frac{\partial\varphi_3}{\partial z_6}(z),\frac{\partial\varphi_4}{\partial z_6}(z)),(1,1,1,1,0,0,0,0)\}$ *is not a free family on* $\mathbb{C}^8$, *for all* $z\in\mathbb{C}^6$.

(iv) *For* $n=5$. *We have* $K(z)=\{(\overline{\lambda_{11}},\overline{\lambda_{21}},\overline{\lambda_{31}},\overline{\lambda_{41}},\frac{\partial\varphi_1}{\partial z_1}(z),\frac{\partial\varphi_2}{\partial z_1}(z),\frac{\partial\varphi_3}{\partial z_1}(z),\frac{\partial\varphi_4}{\partial z_1}(z)),...,$ $(\overline{\lambda_{15}},\overline{\lambda_{25}},\overline{\lambda_{35}},\overline{\lambda_{45}},\frac{\partial\varphi_1}{\partial z_5}(z),\frac{\partial\varphi_2}{\partial z_5}(z),\frac{\partial\varphi_3}{\partial z_5}(z),\frac{\partial\varphi_4}{\partial z_5}(z))\}$ *is a free family on* $\mathbb{C}^8$, *for all* $z\in\mathbb{C}^5$ *and* $\{(\overline{\lambda_{11}},\overline{\lambda_{21}},\overline{\lambda_{31}},\overline{\lambda_{41}},\frac{\partial\varphi_1}{\partial z_1}(z),\frac{\partial\varphi_2}{\partial z_1}(z),\frac{\partial\varphi_3}{\partial z_1}(z),\frac{\partial\varphi_4}{\partial z_1}(z)),...,$ $(\overline{\lambda_{15}},\overline{\lambda_{25}},\overline{\lambda_{35}},\overline{\lambda_{45}},\frac{\partial\varphi_1}{\partial z_5}(z),\frac{\partial\varphi_2}{\partial z_5}(z),\frac{\partial\varphi_3}{\partial z_5}(z),\frac{\partial\varphi_4}{\partial z_5}(z)),(1,1,1,1,0,0,0,0)\}$ *is not a free family on* $\mathbb{C}^8$, *for all* $z\in\mathbb{C}^5$.

(v) *For* $n=4$. *We have* $K(z)=\{(\overline{\lambda_{11}},\overline{\lambda_{21}},\overline{\lambda_{31}},\overline{\lambda_{41}},\frac{\partial\varphi_1}{\partial z_1}(z),\frac{\partial\varphi_2}{\partial z_1}(z),\frac{\partial\varphi_3}{\partial z_1}(z),\frac{\partial\varphi_4}{\partial z_1}(z)),...,$ $(\overline{\lambda_{14}},\overline{\lambda_{24}},\overline{\lambda_{34}},\overline{\lambda_{44}},\frac{\partial\varphi_1}{\partial z_4}(z),\frac{\partial\varphi_2}{\partial z_4}(z),\frac{\partial\varphi_3}{\partial z_4}(z),\frac{\partial\varphi_4}{\partial z_4}(z))\}$ *is a free family on* $\mathbb{C}^8$, *for all* $z\in\mathbb{C}^4$ *and* $K(z)\cup\{(1,1,1,1,0,0,0,0)\}$ *is not a free family on* $\mathbb{C}^8$, *for all* $z\in\mathbb{C}^4$.

(vi) $n\in\{2,3\}$. *We have the same conclusion described as above.*

(vii) $n=1$. $\varphi_1,\varphi_2,\varphi_3,\varphi_4$ *are constant functions on* $\mathbb{C}$ *and* ($\lambda_1\neq 0$, *or* $\lambda_2\neq 0$, *or* $\lambda_3\neq 0$, *or* $\lambda_4\neq 0$).

Proof. (A) implies (B). Define

$$\begin{cases}\psi_1(z,w)=\sum_{j=1}^{4}|w-<z/\lambda_j>-\mu_j|^2+\sum_{j=1}^{4}|\varphi_j(z)|^2,\\ \psi_2(z,w)=\sum_{j=1}^{4}|w-\overline{<z/\lambda_j>}-\overline{\mu_j}|^2+\sum_{j=1}^{4}|\varphi_j(z)|^2,\\ \psi_3(z,w)=|2w|^2+\sum_{j=1}^{4}|<z/\lambda_j>+\mu_j|^2+\sum_{j=1}^{4}|\varphi_j(z)|^2\end{cases}$$

for $(z,w)\in\mathbb{C}^n\times\mathbb{C}$. Here $\varphi_j:\mathbb{C}^n\to\mathbb{C}$, φ_j is analytic on $\mathbb{C}^n$ and $|\varphi_j|$ is convex on $\mathbb{C}^n$, $(1\leq j\leq 4)$ and we have the holomorphic representation

$$\begin{cases}f_j(z)=A_j(<z/\lambda_j>+\mu_j)+\overline{B_j}\varphi_j(z)\\ g_j(z)=B_j(<z/\lambda_j>+\mu_j)-\overline{A_j}\varphi_j(z)\end{cases}$$

for all $z\in\mathbb{C}^n$, with $\lambda_j\in\mathbb{C}^n$, $\mu_j\in\mathbb{C}$, $\lambda_j=(\lambda_{j1},...,\lambda_{jn})$, $1\leq j\leq 4$, by Abidi [3, Theorem 1]. Note that $u_j,v_j,u,v,\varphi,\psi_1,\psi_2,\psi_3$ are functions of class C^∞ on $\mathbb{C}^n\times\mathbb{C}$ (for $1\leq j\leq 4$). We have φ is strictly psh on $\mathbb{C}^n\times\mathbb{C}$ if and only if v is strictly psh on $\mathbb{C}^n\times\mathbb{C}$. But v is strictly psh on $\mathbb{C}^n\times\mathbb{C}$ if and only if ψ_2 is strictly psh on $\mathbb{C}^n\times\mathbb{C}$. By lemma 4.2, ψ_2 is strictly psh on $\mathbb{C}^n\times\mathbb{C}$ if and only if ψ_3 is strictly psh on $\mathbb{C}^n\times\mathbb{C}$. Moreover, also by lemma 4.2, we have $n\leq 8$. Observe that u is not strictly psh in all open balls of $\mathbb{C}^n \times \mathbb{C}$ if and only if ψ_1 is not strictly psh in all non- empty open balls of $\mathbb{C}^n \times \mathbb{C}$.

State 1. $n = 8$.

Note that ψ_3 is strictly psh on $\mathbb{C}^8 \times \mathbb{C}$ if and only if ψ_4 is strictly psh on $\mathbb{C}^8$, where $\psi_4(z, w) = \sum_{j=1}^{4} | < z/\lambda_j > +\mu_j|^2 + \sum_{j=1}^{4} |\varphi_j(z)|^2$, for $z \in \mathbb{C}^8$. ψ_4 is a function of class C^∞ on $\mathbb{C}^8$. The Levi Hermitian form of ψ_4 is

$$L(\psi_4)(z)(\alpha) = \sum_{j=1}^{4} | < \alpha/\lambda_j > |^2 + |\sum_{j=1}^{8} \frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j|^2 + |\sum_{j=1}^{8} \frac{\partial \varphi_2}{\partial z_j}(z)\alpha_j|^2 +$$

$$|\sum_{j=1}^{8} \frac{\partial \varphi_3}{\partial z_j}(z)\alpha_j|^2 + |\sum_{j=1}^{8} \frac{\partial \varphi_4}{\partial z_j}(z)\alpha_j|^2 > 0$$

for any $z = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \in \mathbb{C}^8$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8) \in \mathbb{C}^8 \backslash \{0\}$. Therefore the condition $L(\psi_4)(z)(\alpha) = 0$ if and only if $\alpha = 0$. But $L(\psi_4)(z)(\alpha) = 0$ if and only if

$$\begin{cases} < \alpha/\lambda_1 >= 0, \\ < \alpha/\lambda_2 >= 0, \\ < \alpha/\lambda_3 >= 0, \\ < \alpha/\lambda_4 >= 0, \\ < \alpha/(\overline{\frac{\partial \varphi_1}{\partial z_1}}(z), ..., \overline{\frac{\partial \varphi_1}{\partial z_8}}(z)) >= 0, \\ < \alpha/(\overline{\frac{\partial \varphi_2}{\partial z_1}}(z), ..., \overline{\frac{\partial \varphi_2}{\partial z_8}}(z)) >= 0, \\ < \alpha/(\overline{\frac{\partial \varphi_3}{\partial z_1}}(z), ..., \overline{\frac{\partial \varphi_3}{\partial z_8}}(z)) >= 0, \text{ and} \\ < \alpha/(\overline{\frac{\partial \varphi_4}{\partial z_1}}(z), ..., \overline{\frac{\partial \varphi_4}{\partial z_8}}(z)) >= 0, \end{cases}$$

which imply that $\alpha = 0$. Thus, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, (\overline{\frac{\partial \varphi_1}{\partial z_1}}(z), ..., \overline{\frac{\partial \varphi_1}{\partial z_8}}(z)), (\overline{\frac{\partial \varphi_2}{\partial z_1}}(z), ..., \overline{\frac{\partial \varphi_2}{\partial z_8}}(z)),$ $(\overline{\frac{\partial \varphi_3}{\partial z_1}}(z), ..., \overline{\frac{\partial \varphi_3}{\partial z_8}}(z)), (\overline{\frac{\partial \varphi_4}{\partial z_1}}(z), ..., \overline{\frac{\partial \varphi_4}{\partial z_8}}(z)))$ is a basis of the complex vector space $\mathbb{C}^8$, for all $z \in \mathbb{C}^8$. Note that $\mathbb{C}^8$ is considered a complex vector space of dimension 8. Now $\psi_1(z, w) = \sum_{j=1}^{4} |w - < z/\lambda_j > -\mu_j|^2 + \sum_{j=1}^{4} |\varphi_j(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^9$. By Lemma 4.2, ψ_1 is not strictly psh at all points of $\mathbb{C}^8 \times \mathbb{C}$.

State 2. $n = 7$.

The Levi Hermitian form of ψ_3 is

$$L(\psi_3)(z, w)(\alpha, \beta) = 4|\beta|^2 + | < \alpha/\lambda_1 > |^2 + | < \alpha/\lambda_2 > |^2 + | < \alpha/\lambda_3 > |^2 + | < \alpha/\lambda_4 > |^2 +$$

$$|\sum_{j=1}^{7} \frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j|^2 + |\sum_{j=1}^{7} \frac{\partial \varphi_2}{\partial z_j}(z)\alpha_j|^2 + |\sum_{j=1}^{7} \frac{\partial \varphi_3}{\partial z_j}(z)\alpha_j|^2 + |\sum_{j=1}^{7} \frac{\partial \varphi_4}{\partial z_j}(z)\alpha_j|^2,$$

$z=(z_1,...,z_7)\in\mathbb{C}^7$, $\alpha=(\alpha_1,...,\alpha_7)\in\mathbb{C}^7$, $\beta\in\mathbb{C}$. Fix now $z=(z_1,...,z_7)\in\mathbb{C}^7$ and $w\in\mathbb{C}$. We have $L(\psi_3)(z,w)(\alpha,\beta)=0$, implies that $\beta=0$ and

$$\begin{cases} <\alpha/\lambda_1>=0,<\alpha/_2>=0,<\alpha/\lambda_3>=0,<\alpha/\lambda_4>=0,\\ <\alpha/(\frac{\partial\varphi_1}{\partial z_1}(z),...,\frac{\partial\varphi_1}{\partial z_7}(z))>=0,<\alpha/(\frac{\partial\varphi_2}{\partial z_1}(z),...,\frac{\partial\varphi_2}{\partial z_7}(z))>=0,\\ <\alpha/(\frac{\partial\varphi_3}{\partial z_1}(z),...,\frac{\partial\varphi_3}{\partial z_7}(z))>=0,<\alpha/(\frac{\partial\varphi_4}{\partial z_1}(z),...,\frac{\partial\varphi_4}{\partial z_7}(z))>=0 \end{cases}$$

for all $\alpha=(\alpha_1,...,\alpha_7)\in\mathbb{C}^7$, $\beta\in\mathbb{C}$. Moreover, $L(\psi_3)(z,w)(\alpha,\beta)=0$ implies that $\alpha=0\in\mathbb{C}^7$ and $\beta=0$. Indeed,

$$\begin{cases} \alpha_1\overline{\lambda_{11}}+\alpha_2\overline{\lambda_{12}}+...+\alpha_7\overline{\lambda_{17}}=0\\ \alpha_1\overline{\lambda_{21}}+\alpha_2\overline{\lambda_{22}}+...+\alpha_7\overline{\lambda_{27}}=0\\ \alpha_1\overline{\lambda_{31}}+\alpha_2\overline{\lambda_{32}}+...+\alpha_7\overline{\lambda_{37}}=0\\ \alpha_1\overline{\lambda_{41}}+\alpha_2\overline{\lambda_{42}}+...+\alpha_7\overline{\lambda_{47}}=0\\ \alpha_1\frac{\partial\varphi_1}{\partial z_1}(z)+\alpha_2\frac{\partial\varphi_1}{\partial z_2}(z)+...+\alpha_7\frac{\partial\varphi_1}{\partial z_7}(z)=0\\ \alpha_1\frac{\partial\varphi_2}{\partial z_1}(z)+\alpha_2\frac{\partial\varphi_2}{\partial z_2}(z)+...+\alpha_7\frac{\partial\varphi_2}{\partial z_7}(z)=0\\ \alpha_1\frac{\partial\varphi_3}{\partial z_1}(z)+\alpha_2\frac{\partial\varphi_3}{\partial z_2}(z)+...+\alpha_7\frac{\partial\varphi_3}{\partial z_7}(z)=0\\ \alpha_1\frac{\partial\varphi_4}{\partial z_1}(z)+\alpha_2\frac{\partial\varphi_4}{\partial z_2}(z)+...+\alpha_7\frac{\partial\varphi_4}{\partial z_7}(z)=0 \end{cases}$$

and $\alpha=(\alpha_1,\alpha_2,...,\alpha_7)\in\mathbb{C}^7$ implies that $\alpha=0$. This condition is in fact equivalent with

$$\alpha_1(\overline{\lambda_{11}},\overline{\lambda_{21}},\overline{\lambda_{31}},\overline{\lambda_{41}},\frac{\partial\varphi_1}{\partial z_1}(z),\frac{\partial\varphi_2}{\partial z_1}(z),\frac{\partial\varphi_3}{\partial z_1}(z),\frac{\partial\varphi_4}{\partial z_1}(z))+\dots$$
$$+\alpha_7(\overline{\lambda_{17}},\overline{\lambda_{27}},\overline{\lambda_{37}},\overline{\lambda_{47}},\frac{\partial\varphi_1}{\partial z_7}(z),\frac{\partial\varphi_2}{\partial z_7}(z),\frac{\partial\varphi_3}{\partial z_7}(z),\frac{\partial\varphi_4}{\partial z_7}(z))=0\in\mathbb{C}^8$$

implies that $\alpha_1=\ldots=\alpha_7=0$. Thus,

$$\{(\overline{\lambda_{11}},\overline{\lambda_{21}},\overline{\lambda_{31}},\overline{\lambda_{41}},\frac{\partial\varphi_1}{\partial z_1}(z),\frac{\partial\varphi_2}{\partial z_1}(z),\frac{\partial\varphi_3}{\partial z_1}(z),\frac{\partial\varphi_4}{\partial z_1}(z)),...,$$
$$(\overline{\lambda_{17}},\overline{\lambda_{27}},\overline{\lambda_{37}},\overline{\lambda_{47}},\frac{\partial\varphi_1}{\partial z_7}(z),\frac{\partial\varphi_2}{\partial z_7}(z),\frac{\partial\varphi_3}{\partial z_7}(z),\frac{\partial\varphi_4}{\partial z_7}(z))\}$$

is a free family of 7 vectors of $\mathbb{C}^8$, $\mathbb{C}^8$ is a complex vector space of dimension 8. It follows that there exists $a=(a_1,...,a_8)\in\mathbb{C}^8\backslash\{0\}$ such that

$$((\overline{\lambda_{11}},\overline{\lambda_{21}},\overline{\lambda_{31}},\overline{\lambda_{41}},\frac{\partial\varphi_1}{\partial z_1}(z),\frac{\partial\varphi_2}{\partial z_1}(z),\frac{\partial\varphi_3}{\partial z_1}(z),\frac{\partial\varphi_4}{\partial z_1}(z)),...,$$
$$(\overline{\lambda_{17}},\overline{\lambda_{27}},\overline{\lambda_{37}},\overline{\lambda_{47}},\frac{\partial\varphi_1}{\partial z_7}(z),\frac{\partial\varphi_2}{\partial z_7}(z),\frac{\partial\varphi_3}{\partial z_7}(z),\frac{\partial\varphi_4}{\partial z_7}(z)),a)$$

is a basis of the complex vector space $\mathbb{C}^8$. The Levi Hermitian form of ψ_1 is

$$L(\psi_1)(z,w)(\alpha,\beta)=\sum_{j=1}^{4}|\beta-<\alpha/\lambda_j>|^2+\sum_{k=1}^{4}|\sum_{j=1}^{7}\frac{\partial\varphi_k}{\partial z_j}(z)\alpha_j|^2$$

for $\alpha=(\alpha_1,...,\alpha_7)\in\mathbb{C}^7$ and $\beta\in\mathbb{C}$. We will prove that ψ_1 is not strictly psh at all points of $\mathbb{C}^7\times\mathbb{C}$. In fact we prove that there exists $(\alpha,\beta)\in(\mathbb{C}^7\backslash\{0\})\times(\mathbb{C}\backslash\{0\})$ such that $L(\psi_1)(z,w)(\alpha,\beta)=0$. Moreover, $L(\psi_1)(z,w)(\alpha,\beta)=0$ if and only if

$$\begin{cases} <\alpha/\lambda_1>-\beta=0,<\alpha/\lambda_2>-\beta=0,<\alpha/\lambda_3>-\beta=0,\\ <\alpha/\lambda_4>-\beta=0,\sum_{j=1}^{7}\frac{\partial\varphi_1}{\partial z_j}(z)\alpha_j=0,\sum_{j=1}^{7}\frac{\partial\varphi_2}{\partial z_j}(z)\alpha_j=0,\\ \sum_{j=1}^{7}\frac{\partial\varphi_3}{\partial z_j}(z)\alpha_j=0,\sum_{j=1}^{7}\frac{\partial\varphi_4}{\partial z_j}(z)\alpha_j=0 \end{cases}$$

where $\alpha = (\alpha_1, ..., \alpha_7) \in \mathbb{C}^7$ and $\beta \in \mathbb{C}$. Then

$$\begin{cases} \alpha_1\overline{\lambda_{11}} + \alpha_2\overline{\lambda_{12}} + ... + \alpha_7\overline{\lambda_{17}} - \beta = 0 \\ \alpha_1\overline{\lambda_{21}} + \alpha_2\overline{\lambda_{22}} + ... + \alpha_7\overline{\lambda_{27}} - \beta = 0 \\ \alpha_1\overline{\lambda_{31}} + \alpha_2\overline{\lambda_{32}} + ... + \alpha_7\overline{\lambda_{37}} - \beta = 0 \\ \alpha_1\overline{\lambda_{41}} + \alpha_2\overline{\lambda_{42}} + ... + \alpha_7\overline{\lambda_{47}} - \beta = 0 \\ \alpha_1\frac{\partial\varphi_1}{\partial z_1}(z) + \alpha_2\frac{\partial\varphi_1}{\partial z_2}(z) + ... + \alpha_7\frac{\partial\varphi_1}{\partial z_7}(z) = 0 \\ \alpha_1\frac{\partial\varphi_2}{\partial z_1}(z) + \alpha_2\frac{\partial\varphi_2}{\partial z_2}(z) + ... + \alpha_7\frac{\partial\varphi_2}{\partial z_7}(z) = 0 \\ \alpha_1\frac{\partial\varphi_3}{\partial z_1}(z) + \alpha_2\frac{\partial\varphi_3}{\partial z_2}(z) + ... + \alpha_7\frac{\partial\varphi_3}{\partial z_7}(z) = 0 \\ \alpha_1\frac{\partial\varphi_4}{\partial z_1}(z) + \alpha_2\frac{\partial\varphi_4}{\partial z_2}(z) + ... + \alpha_7\frac{\partial\varphi_4}{\partial z_7}(z) = 0. \end{cases}$$

Thus

$$\alpha_1(\overline{\lambda_{11}}, \overline{\lambda_{21}}, \overline{\lambda_{31}}, \overline{\lambda_{41}}, \frac{\partial\varphi_1}{\partial z_1}(z), \frac{\partial\varphi_2}{\partial z_1}(z), \frac{\partial\varphi_3}{\partial z_1}(z), \frac{\partial\varphi_4}{\partial z_1}(z)) + ... + \alpha_7(\overline{\lambda_{17}}, \overline{\lambda_{27}},$$

$$\overline{\lambda_{37}}, \overline{\lambda_{47}}, \frac{\partial\varphi_1}{\partial z_7}(z), \frac{\partial\varphi_2}{\partial z_7}(z), \frac{\partial\varphi_3}{\partial z_7}(z), \frac{\partial\varphi_4}{\partial z_7}(z)) - \beta(1,1,1,1,0,0,0,0) = 0 \in \mathbb{C}^8.$$

Now if

$$K(z) = ((\overline{\lambda_{11}}, \overline{\lambda_{21}}, \overline{\lambda_{31}}, \overline{\lambda_{41}}, \frac{\partial\varphi_1}{\partial z_1}(z), \frac{\partial\varphi_2}{\partial z_1}(z), \frac{\partial\varphi_3}{\partial z_1}(z), \frac{\partial\varphi_4}{\partial z_1}(z)), ...,$$
$$(\overline{\lambda_{17}}, \overline{\lambda_{27}}, \overline{\lambda_{37}}, \overline{\lambda_{47}}, \frac{\partial\varphi_1}{\partial z_7}(z), \frac{\partial\varphi_2}{\partial z_7}(z), \frac{\partial\varphi_3}{\partial z_7}(z), \frac{\partial\varphi_4}{\partial z_7}(z)), (1,1,1,1,0,0,0,0))$$

is a basis of the complex vector space $\mathbb{C}^8$, then $\alpha_1 = ... = \alpha_7 = \beta = 0$. Then there exists $R > 0$ such that for all $\xi \in B(z, R) \subset \mathbb{C}^7$, $K(\xi)$ is a basis of $\mathbb{C}^8$ by using the determinant $\det(K(z)) \neq 0$. Since $K(\xi)$ is a basis of $\mathbb{C}^8$, the function ψ_1 is strictly psh on $B(z, R) \times \mathbb{C}$. Now since ψ_1 is not strictly psh in all open balls of $\mathbb{C}^7 \times \mathbb{C}$, we have a contradiction. Consequently, $K(z)$ is not a basis of $\mathbb{C}^8$. Then there exists $(\alpha_1, ..., \alpha_7, \beta) \in \mathbb{C}^7 \times \mathbb{C}\backslash\{0\}$ such that

$$\alpha_1(\overline{\lambda_{11}}, \overline{\lambda_{21}}, \overline{\lambda_{31}}, \overline{\lambda_{41}}, \frac{\partial\varphi_1}{\partial z_1}(z), \frac{\partial\varphi_2}{\partial z_1}(z), \frac{\partial\varphi_3}{\partial z_1}(z), \frac{\partial\varphi_4}{\partial z_1}(z)) + \ldots + \alpha_7(\overline{\lambda_{17}}, \overline{\lambda_{27}},$$

$$\overline{\lambda_{37}}, \overline{\lambda_{47}}, \frac{\partial\varphi_1}{\partial z_7}(z), \frac{\partial\varphi_2}{\partial z_7}(z), \frac{\partial\varphi_3}{\partial z_7}(z), \frac{\partial\varphi_4}{\partial z_7}(z)) - \beta(1,1,1,1,0,0,0,0) = 0 \in \mathbb{C}^8.$$

If now $\beta = 0$, then $\alpha = 0 \in \mathbb{C}^7$, because

$$\{(\overline{\lambda_{11}}, \overline{\lambda_{21}}, \overline{\lambda_{31}}, \overline{\lambda_{41}}, \frac{\partial\varphi_1}{\partial z_1}(z), \frac{\partial\varphi_2}{\partial z_1}(z), \frac{\partial\varphi_3}{\partial z_1}(z), \frac{\partial\varphi_4}{\partial z_1}(z)), ..., (\overline{\lambda_{17}}, \overline{\lambda_{27}},$$

$$\overline{\lambda_{37}}, \overline{\lambda_{47}}, \frac{\partial\varphi_1}{\partial z_7}(z), \frac{\partial\varphi_2}{\partial z_7}(z), \frac{\partial\varphi_3}{\partial z_7}(z), \frac{\partial\varphi_4}{\partial z_7}(z))\}$$

is a free family of 7 vectors of the complex vector space $\mathbb{C}^8$, a contradiction. Thus $\beta \neq 0$. Since $\beta \neq 0$, $\alpha \neq 0$. Consequently, there exists $(\alpha, \beta) \in (\mathbb{C}^7\backslash 0\}) \times (\mathbb{C}\backslash\{0\}) \subset \mathbb{C}^8\backslash\{0\}$ such that $L(\psi_1)(z, w)(\alpha, \beta) = 0$, for all $(z, w) \in \mathbb{C}^7 \times \mathbb{C}$. ψ_1 is not strictly psh at all points of $\mathbb{C}^7 \times \mathbb{C}$. Note that the proof of the other cases are obvious by the above proof. □

Using the notation of theorem 4.9, we can study several problems, for example the following questions.

Question 4.10. Find all the holomorphic functions $f_1, g_1, f_2, g_2, f_3, g_3, f_4, g_4$ such that $u_1+u_2+u_3$ is not strictly psh in all open balls of $\mathbb{C}^n\times\mathbb{C}$. v_1+v_2 and v_3+v_4 are not strictly psh in all open balls of $\mathbb{C}^n\times\mathbb{C}$, but φ is strictly psh on $\mathbb{C}^n\times\mathbb{C}$ (u_1, u_2, u_3 and u_4 are convex functions on $\mathbb{C}^n\times\mathbb{C}$).

Question 4.11. Characterize exactly all the holomorphic functions f_j, g_j, $1\le j\le 4$ such that u, $v_1+v_2+v_3$, $v_1+v_2+v_4$, $v_1+v_3+v_4$ and $v_2+v_3+v_4$ are functions not strictly psh in all open balls of $\mathbb{C}^n\times\mathbb{C}$, but v is strictly psh on $\mathbb{C}^n\times\mathbb{C}$, u_1, u_2, u_3 and u_4 are convex functions on $\mathbb{C}^n\times\mathbb{C}$. Now find all the holomorphic functions f_j, g_j $(1\le j\le 4)$, such that $(u+v_1+v_2+v_3)$ is not strictly psh on all open balls of $\mathbb{C}^n\times\mathbb{C}$, but φ is strictly psh on $\mathbb{C}^n\times\mathbb{C}$ and φ is not strictly convex on all not empty open balls of $\mathbb{C}^n\times\mathbb{C}$, u_1, u_2, u_3, u_4 are convex functions on $\mathbb{C}^n\times\mathbb{C}$.

We can generalize the above two questions for $2N$ functions $f_j, g_j, 1\le j\le N$ and we obtain several classifications of many classes of holomorphic functions.

5 Concluding remarks

Theorem 5.1. *Let $g:\mathbb{C}^n\to\mathbb{C}$ be an analytic function. Given $a, c\in\mathbb{C}^n$, $b, d\in\mathbb{C}$ and define $u(z)=|g(z)+<z/a>+b|^2$, $v(z)=|g(z)+<z/c>+d|^2$, $z\in\mathbb{C}^n$. Assume that u and v are convex functions on $\mathbb{C}^n$ and there exists $z^0\in\mathbb{C}^n$ such that $(<z^0/a>+b)\neq(<z^0/c>+d)$. Then g is an analytic polynomial on $\mathbb{C}^n$ with $deg(g)\le 2$.*

Proof. Case 1. $n=1$.

$u(z)=|g(z)+\overline{a}z+b|^2$, $v(z)=|g(z)+\overline{c}z+d|^2$, for $z\in\mathbb{C}$ and we have $(a,b)\neq(c,d)$. Observe that f_1 and f_2 are holomorphic functions on $\mathbb{C}$, $f_1(z)=g(z)+\overline{a}z+b$, $f_2(z)=g(z)+\overline{c}z+d$, $z\in\mathbb{C}$. Since $|f_1|^2$ is convex on $\mathbb{C}$, by [2, Théorème 19], the function $|f_1'|^2$ is convex on $\mathbb{C}$. Thus $|g'+\overline{a}|$ is convex on $\mathbb{C}$. Also $|g'+\overline{c}|$ is convex on $\mathbb{C}$. Assume that $a=c$. Then $b\neq d$. Define $g_1(z)=g(z)+\overline{a}z$, for $z\in\mathbb{C}$. In this case, we have $|g_1+b|^2$ and $|g_1+c|^2$ are convex functions on $\mathbb{C}$. By theorems 3.1 and 3.7, we have g_1 is affine on $\mathbb{C}$. Consequently, g is affine on $\mathbb{C}$. Asume that $a\neq c$. Since $|g'+\overline{a}|$ and $|g'+\overline{c}|$ are convex functions on $\mathbb{C}$, g' is affine on $\mathbb{C}$. Therefore g is a holomorphic polynomial on $\mathbb{C}$ with $\deg(g)\le 2$.

Case 2. $n\ge 2$. This is obvious by the problem of fibration. □

Corollary 5.2. *Here we use the notations of theorem 5.1. Assume that $b\neq d$ and $a=c$. Then g is affine on $\mathbb{C}^n$.*

We can use theorem 5.1 for the study of the following problem. Let $n\ge 1$. Find all the holomorphic functions $g:\mathbb{C}^n\to\mathbb{C}$ such that $|g+\varphi_1|^2$ and $|g+\varphi_2|^2$ are convex functions on $\mathbb{C}^n$, where $\varphi_1,\varphi_2:\mathbb{C}^n\to\mathbb{C}$ are two holomorphic functions such that $|\varphi_1|^2$ and $|\varphi_2|^2$ are convex functions on $\mathbb{C}^n$.

Remark 5.3. Let $a_1, a_2\in\mathbb{C}^m$, $m\ge 1$. Let $f_1, f_2:\mathbb{C}^m\to\mathbb{C}$ be two analytic functions, $n\ge 1$. Define

$$\begin{cases} u(z,w)=|<w/a_1>-f_1(z)|^2+|<w/a_2>-f_2(z)|^2,\\ v(z,w)=|<w/a_1>-\overline{f_1}(z)|^2+|<w/a_2>-\overline{f_2}(z)|^2\end{cases}$$

for $(z,w)\in\mathbb{C}^n\times\mathbb{C}^m$. (A) We can study the following two problems

$$\begin{cases} u \text{ is convex on } \mathbb{C}^n\times\mathbb{C}^m,\\ v \text{ is strictly psh on } \mathbb{C}^n\times\mathbb{C}^m\end{cases}$$

and

$$\begin{cases} u \text{ is convex on } \mathbb{C}^n\times\mathbb{C}^m,\\ v \text{ is strictly psh but not strictly convex on } \mathbb{C}^n\times\mathbb{C}^m.\end{cases}$$

(B) Assume that $\{a_1, a_2\}$ is a free family on $\mathbb{C}^m$ and let $f_3 : \mathbb{C}^n \to \mathbb{C}$ be a analytic function. Define

$$\begin{cases} \varphi(z,w) = u(z,w) + |f_3(z)|^2, \\ \psi(z,w) = v(z,w) + |f_3(z)|^2 \end{cases}$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$. We prove that φ is convex on $\mathbb{C}^n \times \mathbb{C}^m$ if and only if f_1, f_2 are affine functions on $\mathbb{C}^n$, $|f_3|$ is convex on $\mathbb{C}^n$ and we can study the question where

$$\begin{cases} \varphi \text{ is convex but not strictly psh on } \mathbb{C}^n \times \mathbb{C}^m, \\ \psi \text{ is strictly psh on } \mathbb{C}^n \times \mathbb{C}^m. \end{cases}$$

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