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On the Four-Parameter T-extended Standard U-quadratic Exponentiated Weibull Distribution

Idzhar A. Lakibul[1] and Bernadette F. Tubo[2,⋆]

[1,2]Department of Mathematics and Statistics
MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines
idzhar.lakibul@g.msuiit.edu.ph, bernadette.tubo@g.msuiit.edu.ph

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Abstract

This paper identifies the characteristics of a three-parameter distribution which is an extension of the Exponentiated Weibull distribution (ExW) introduced by Pal, et al. in 2006 [18] and Mudholkar and Srivastava in 1993 [15]. This extended distribution is labeled as T-extended Standard U-quadratic Exponentiated Weibull distribution ($TeSU$-ExW). The proposed distribution uses the properties of the U-quadratic distribution and the Transformed Transformer family of distributions in its derivation. Let X be a random variable that follows an ExW distribution with scale parameter λ and shape parameters β and γ. This study derives the cumulative distribution function (*cdf*) and the corresponding probability density function (*pdf*) of $TeSU$-ExW. Moreover, properties such as its moment, moment generating function, mean, median, mode and variance are also obtained in this paper.

1 Introduction

Statistical distributions are important in modelling and describing the behavior of real lifetime data. The choice of a probability distribution function is important specifically in the field of survival analysis or reliability analysis in engineering, since most of the behavior of the data in these fields are complex which cannot be easily fitted using the classical probability distribution function.

Azzalini (1985) introduced a skewed family of distribution for generating a distribution with additional skewed parameter [5]. Other identified family of distributions are the Marshall-Olkin extended (MOE) family [14], and the exponentiated family of distributions [11]. Moreover, several authors introduced composite distributions by combining two or more known competing distributions through transformations, like the Kumaraswamy-G (Kw-G) family [7], the Mc Donald-G family [1], the Exponentiated transformed transformer family [4], the Exponentiated Generalized family [8], the Kumarsway Marshall- Olkin-G family [2], the Generalized odd log-logistic family [6], the Generalized transmuted G family [17], the Odd lindley - G family [10] and the Exponentiated kumarasway - G class family [21].

This paper derives an extended distribution of the U- quadratic distribution, Exponentiated -

⋆Corresponding author
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G and the transformed- transformer family of distributions, for short, the *TeSU-ExG* family of distributions and explored a derived model using the Weibull as the baseline distribution. The Weibull distribution is mostly used in life data analysis because of its ability to adapt to different situations. The application of models based on the Weibull distribution is also widely used in reliability engineering and various other fields to model the distribution of time until an event occurs. The distribution is flexible and can take on a variety of shapes, making it suitable for modeling a wide range of real-world situations.
In particular, the *TeSU-ExG* will not only be generating a distribution that can capture the behavior of a lifetime data but will also generate more shapes that can mimic the behavior of the real phenomenon depending on the choice of the baseline distribution, in this case, the Weibull distribution.
This paper focuses only on the derivation of some statistical properties of the *TeSU-ExW* distribution, including moment, mean, variance, moment generating function, median and mode. However, once the mathematical rigor in this research work is done, its application to model real-world situations will be explored.

2 Generating Family of Distributions

This section discusses some continuous distributions that will be used in this paper.

2.1 U-quadratic

The most basic distribution considered in this paper is the U-quadratic distribution. This distribution is a useful model for symmetric bimodal processes. The cumulative distribution and probability density functions are defined in Definition 2.1 .

Definition 2.1. Let T be a random variable that follows a U-quadratic distribution. Then the *cdf* of T is given by

$$L(t) = \frac{m}{3}\left[(t-n)^3 + (n-a)^3\right]$$

with corresponding *pdf*

$$l(t) = m(t-n)^2, \tag{1}$$

where $t \in [a,b], a < b \in \mathbb{R}, m = \dfrac{12}{(b-a)^3}$ and $n = \dfrac{a+b}{2}$.

The next subsection shows the derivation of the extended Standard U-quadratic (eSU) distribution.

2.2 extended Standard U-quadratic

Consider the special case of the T-X family which was introduced by [3]. Accordingly, for any arbitrary baseline cumulative distribution function (*cdf*) $G(x)$, a new cdf $F(x)$ can be generated using the equation

$$F(x) = \int_0^{G(x)} f(t)dt \tag{2}$$

where $f(t)$ is a probability density function (*pdf*) of a random variable T with support on the interval $[0,1]$. Also, consider the Transmuted-G family of distributions introduced by [20], that is, for any baseline cdf $G(x)$, one can define a new cdf $K(x)$ given by

$$K(x) = (1+\lambda)G(x) - \lambda(G(x))^2, \tag{3}$$

where $\lambda \in [-1, 1]$. Note that equation (3) can be written as

$$K(x) = \int_0^{G(x)} f(t)dt$$

where

$$f(t) = 1 + \lambda - 2\lambda t\ I_{[0,1]}(t) = (1-\lambda)f_1(t) + \lambda f_2(t) \tag{4}$$

with $f_1(t) = 1\ I_{[0,1]}(t)$ and $f_2(t) = 2(1-t)\ I_{[0,1]}(t)$. Hence, $f(t)$ can be written as a mixture of two probability density functions with support set on the interval $[0, 1]$.
Now consider the *pdf* of the U-quadratic distribution given in equation (1). To standardize equation (1), let $a = 0$ and $b = 1$. Then, equation (1) becomes

$$l(t) = 12\left(t - \frac{1}{2}\right)^2, \tag{5}$$

where $t \in [0, 1]$. Substituting $l(t)$ of (5) in equation (4) for $f_2(t)$ derives the *pdf* of the eSU-quadratic distribution, given in Definition 2.2.

Definition 2.2. [13] Let T be a random variable that follows a eSU distribution. Then the *pdf* of T is given by

$$f_{eSU}(t) = 1 - \lambda + 3\lambda(2t-1)^2, \tag{6}$$

where $t \in [0, 1]$ and $\lambda \in [-\frac{1}{2}, 1]$.

In addition, the following figures present the graph of the probability density functions (*pdf*) of the U-quadratic distribution, the Standard U-quadratic distribution and the extended Standard U-quadratic distribution. Figure 1 shows that the graph of the *pdf* of the U-quadratic distribution has a bathtub shape. Figure 2 is the special case of the of the U-quadratic distribution for fixed values of $a = 0$ and $b = 1$. Moreover, Figure 3 shows the *pdf* of the extended Standard U-quadratic (eSU) distribution with the following shapes on the given values of λ: (i) if $0 < \lambda \leq 1$ the *pdf* of eSU distribution has bathtub shape; (ii) if $\lambda = 0$ the *pdf* of the eSU distribution has a constant function shape; and (iii) if $-\frac{1}{2} \leq \lambda < 0$ the *pdf* of the eSU distribution become inverted bathtub shape.

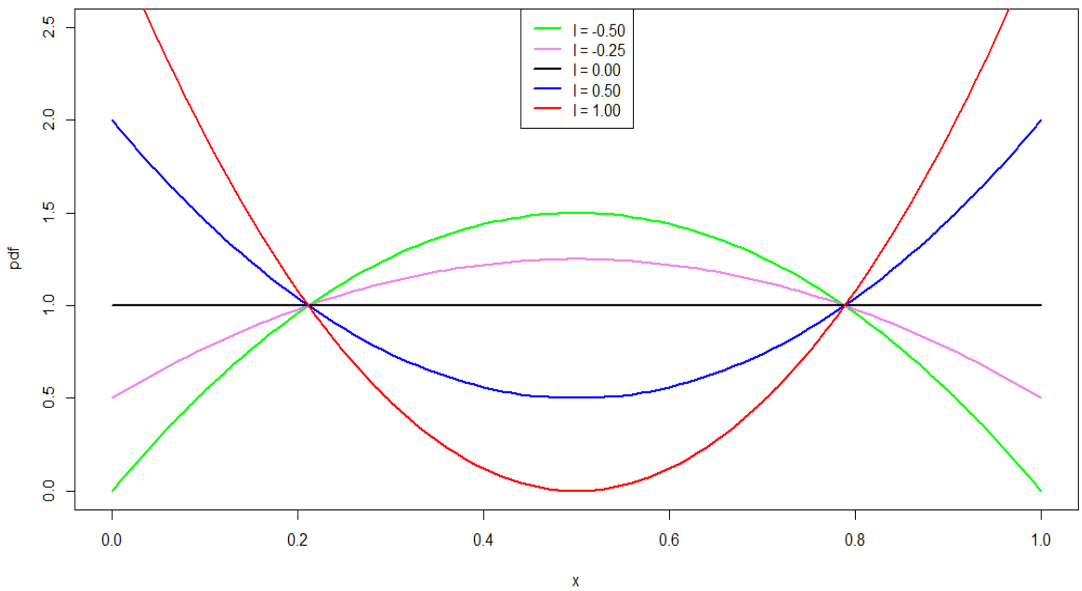

Figure 1. Graph of the *pdf* of the U-quadratic distribution for varied parameter values

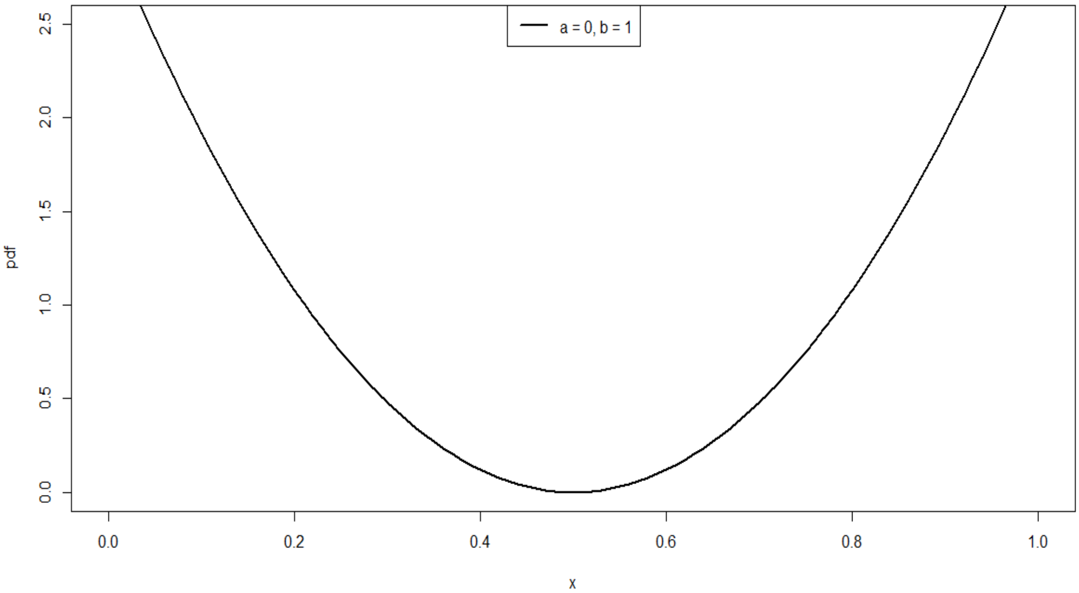

Figure 2. Graph of the *pdf* of the Standard U-quadratic distribution

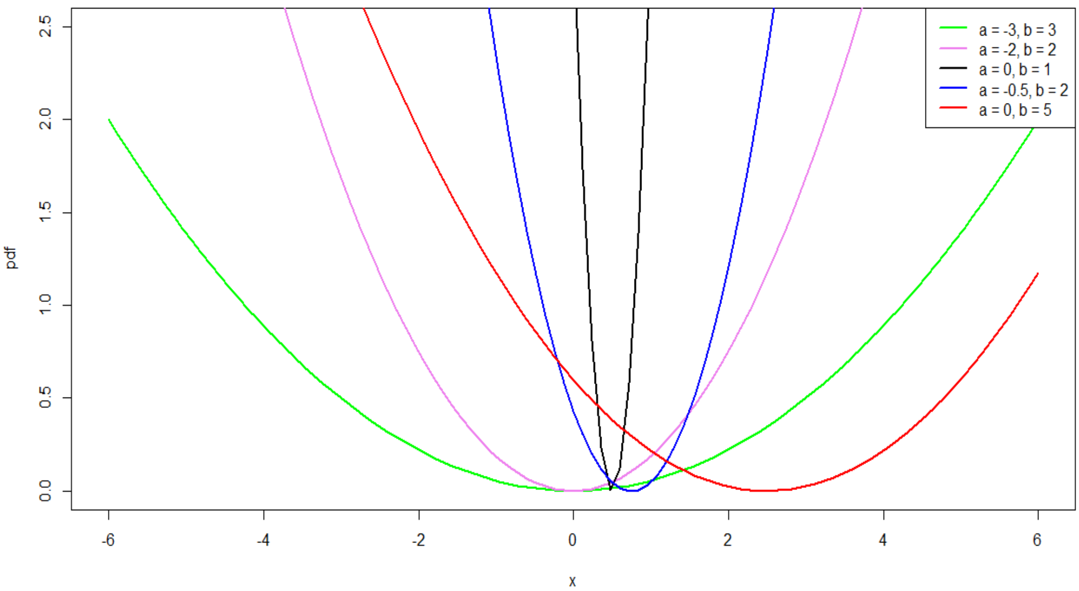

Figure 3. Graph of the *pdf* of the extended Standard U-quadratic distribution

2.3 T-extended Standard U-quadratic-G

This subsection introduces a T-extended Standard U-quadratic ($TeSU$)-G Family of distribution. Using equation (2) and the *pdf* of eSU in (6) derives the *cdf* of TeSU-G family of distribution and is presented in Theorem 2.3.

Theorem 2.3. *[13] Let X be a random variable that follows a $TeSU$-G distribution. Then the cdf and the pdf of $TeSU$-G family of distribution are, respectively,*

$$F_{TeSU\text{-}G}(x) = (1 + 2\lambda)(G(x)) - 6\lambda(G(x))^2 + 4\lambda(G(x))^3$$

and

$$f_{TeSU\text{-}G}(x) = g(x)[1 - \lambda + 3\lambda(2G(x) - 1)^2].$$

Proof. Recall that

$$\begin{aligned} F(x) &= \int_0^{G(x)} f_{eSU}(t)dt \\ &= \int_0^{G(x)} \left[1-\lambda+3\lambda(2t-1)^2\right] dt \\ &= (1+2\lambda)G(x) - 6\lambda\left(G(x)\right)^2 + 4\lambda\left(G(x)\right)^3, \end{aligned}$$

where $x \in \mathbb{R}$. It follows that for $\lambda \in \left[-\frac{1}{2}, 1\right]$, the *cdf* of the TeSU-G distribution is given by

$$F_{TeSU\text{-}G}(x) = (1+2\lambda)(G(x)) - 6\lambda(G(x))^2 + 4\lambda(G(x))^3 \tag{7}$$

with corresponding *pdf*

$$f_{TeSU\text{-}G}(x) = g(x)[1-\lambda+3\lambda(2G(x)-1)^2], \tag{8}$$

where $g(x)$ is the *pdf* associated with a baseline *cdf* $G(x)$. □

Remark 2.4. If $\lambda = 0$, the *cdf* of $TeSU$-G reduces to the *cdf* of the baseline distribution.

2.4 *TeSU*-*E*xponentiated *G*

Consider the Exponentiated-G family of distributions introduced by Gupta [11], that is, for any baseline *cdf* $B(x)$, one can define a new *cdf* $R(x)$ given by

$$R(x) = (B(x))^\rho$$

with corresponding pdf

$$r(x) = \rho b(x)(B(x))^{\rho-1},$$

where $\rho > 0$ and $b(x)$ is the *pdf* of $B(x)$. The *pdf* of the $TeSU$-Exponentiated G Family of distributions ($TeSU$-ExG) is obtained by setting $G(x) = R(x)$ in (8) and is given in Definition 2.5.

Definition 2.5. Let X be a random variable that follows a $TeSU$-ExG distribution. Then the *pdf* of X is given by

$$f_{TeSU\text{-}ExG}(x) = \rho b(x)(B(x))^{\rho-1}[1-\lambda+3\lambda(2(B(x))^\rho-1)^2], x \in \mathbb{R}$$

with corresponding *cdf* given by

$$F_{TeSU\text{-}ExG}(x) = (1+2\lambda)(B(x))^\rho - 6\lambda(B(x))^{2\rho} + 4\lambda(B(x))^{3\rho}, \tag{9}$$

where $\lambda \in [-\frac{1}{2}, 1]$ and $\rho > 0$.

Remark 2.6. The following conditions produces some special cases of the $TeSU$-ExG:

(*i*) If $\rho = 1$ then $TeSU$-ExG reduces to $TeSU$-G;

(*ii*) If $\lambda = 0$ then $TeSU$-ExG reduces to Ex-G; and

(*iii*) If $\lambda = 0$ and $\rho = 1$ then $TeSU$-ExG reduces to the baseline distribution.

3 $TeSU$-W and $TeSU$-ExW Distributions

This section discusses the $TeSU$ and $TeSU$-Exponentiated using the Weibull framework as the baseline distribution.

3.1 $TeSU$-Weibull Distribution

Definition 3.1 gives the *pdf* and *cdf* of the Weibull distribution.

Definition 3.1. [22] A random variable X is said to have a Weibull distribution if it has a cumulative density function (*cdf*) and probability density function (*pdf*) given, respectively, by,

$$F_W(x) = 1 - e^{-\tau x^\beta}, \quad x > 0, \tau > 0 \text{ and } \beta > 0, \tag{10}$$

and

$$f_W(x) = \tau\beta x^{\beta-1} e^{-\tau x^\beta}, \tag{11}$$

where τ and β are the scale and shape parameters.

Theorem 3.2. *[13] Suppose that X follows the Weibull distribution with cdf given in (10) and pdf in (11), then the cdf and pdf of the $TeSU$-Weibull distribution ($TeSU$-W) is*

$$F_{TeSU\text{-}W}(x) = 1 - e^{-\tau x^\beta}(1 + 2\lambda - 6\lambda e^{-\tau x^\beta} + 4\lambda e^{-2\tau x^\beta})$$

and

$$f_{TeSU\text{-}W}(x) = \tau\beta x^{\beta-1} e^{-\tau x^\beta}(1 + 2\lambda - 12\lambda e^{-\tau x^\beta} + 12\lambda e^{-2\lambda e^{-2\tau x^\beta}}),$$

respectively.

Proof. Suppose that a random variable X has Weibull distribution with cdf (10) and pdf (11). Then, the *cdf* of the $TeSU$-Weibull distribution ($TeSU$-W) is derived by substituting (10) in equation (7), so that we have

$$F_{TeSU\text{-}W}(x) = 1 - e^{-\tau x^\beta}(1 + 2\lambda - 6\lambda e^{-\tau x^\beta} + 4\lambda e^{-2\tau x^\beta}), \tag{12}$$

where $\tau > 0$, $\beta > 0$, $\lambda \in [-\frac{1}{2}, 1]$ and $x \geq 0$. It follows also that the *pdf* is given by

$$f_{TeSU\text{-}W}(x) = \tau\beta x^{\beta-1} e^{-\tau x^\beta}(1 + 2\lambda - 12\lambda e^{-\tau x^\beta} + 12\lambda e^{-2\lambda e^{-2\tau x^\beta}}). \tag{13}$$

$\square$

Remark 3.3. [13] The $TeSU$-exponential distribution ($TeSU$-E) is derived as a special case of the $TeSU$-W when $\beta = 1$. Hence, the *cdf* of $TeSU$-E distribution is given as

$$F_{TeSU\text{-}E}(x) = 1 - e^{-\tau x}\left(1 + 2\lambda - 6\lambda e^{-\tau x} + 4\lambda e^{-2\tau x}\right),$$

where $\tau > 0$, $\lambda \in [-\frac{1}{2}, 1]$ and $x \geq 0$ with corresponding *pdf* given by

$$f_{TeSU\text{-}E}(x) = \tau e^{-\tau x}\left(1 + 2\lambda - 12\lambda e^{-\tau x} + 12\lambda e^{-2\lambda e^{-2\tau x}}\right).$$

3.2 $TeSU$-Exponentiated Weibull Distribution

This subsection derives an extended version of the Exponentiated Weibull distribution labeled as the T-extended Standard U-quadratic-Exponentiated Weibull distribution ($TeSU$-ExW).

Theorem 3.4. *Let X be a random variable that follows a $TeSU$-ExW distribution. Then the cdf and pdf of X is given by*

$$F_{TeSU\text{-}ExW}(x) = (1+2\lambda)\left(1-e^{-\tau x^{\beta}}\right)^{\rho} - 6\lambda\left(1-e^{-\tau x^{\beta}}\right)^{2\rho} + 4\lambda\left(1-e^{-\tau x^{\beta}}\right)^{3\rho}$$

and

$$f_{TeSU\text{-}ExW}(x) = \beta\rho\tau x^{\beta-1}e^{-\tau x^{\beta}}\left(1-e^{-\tau x^{\beta}}\right)^{\rho-1}\left\{1-\lambda+3\lambda\left[2(1-e^{-\tau x^{\beta}})^{\rho}-1\right]^{2}\right\},$$

respectively.

Proof. The *cdf* of the $TeSU$-ExW is obtained by inserting (10) into (9) and can be verified to be equal to

$$F_{TeSU\text{-}ExW}(x) = (1+2\lambda)\left(1-e^{-\tau x^{\beta}}\right)^{\rho} - 6\lambda\left(1-e^{-\tau x^{\beta}}\right)^{2\rho} + 4\lambda\left(1-e^{-\tau x^{\beta}}\right)^{3\rho} \tag{14}$$

with corresponding *pdf* given by

$$f_{TeSU\text{-}ExW}(x) = \beta\rho\tau x^{\beta-1}e^{-\tau x^{\beta}}\left(1-e^{-\tau x^{\beta}}\right)^{\rho-1}\left\{1-\lambda+3\lambda\left[2(1-e^{-\tau x^{\beta}})^{\rho}-1\right]^{2}\right\}, \tag{15}$$

where $\lambda \in [-\frac{1}{2}, 1]$, $x > 0$, $\tau > 0$, $\beta > 0$ and $\rho > 0$. □

Remark 3.5. The following are special cases of $TeSU$-ExW:

(*i*) If $\beta = 1$ then $TeSU$-ExW reduces to $TeSU$-Exponentiated Exponential distribution;

(*ii*) If $\lambda = 0$ then $TeSU$-ExW reduces to Exponentiated Weibull distribution;

(*iii*) If $\lambda = 0$ and $\rho = 1$ then $TeSU$-ExW reduces to Weibull distribution;

(*iv*) If $\rho = 1$ and $\beta = 1$ then $TeSU$-ExW reduces to $TeSU$-Exponential distribution;

(*v*) If $\beta = 1$, $\rho = 1$ and $\lambda = 0$ then $TeSU$-ExW reduces to Exponential distribution;

(*vi*) If $\rho = 1$ then $TeSU$-ExW reduces to $TeSU$-Weibull distribution; and

(*vii*) If $\lambda = 0$ and $\beta = 1$ then $TeSU$-ExW reduces to Exponentiated Exponential.

Figure 4 shows the behavior of the *pdf* of the $TeSU$-ExW distribution for $\tau = 1.4$, $\beta = 2.5$, $\rho = 0.5$ and for some values of λ while Figure 5 depicts the trend of the pdf of the $TeSU$-ExW distribution for $\tau = 1.4$, $\beta = 2.5$, $\lambda = 0.5$ and for some values of ρ

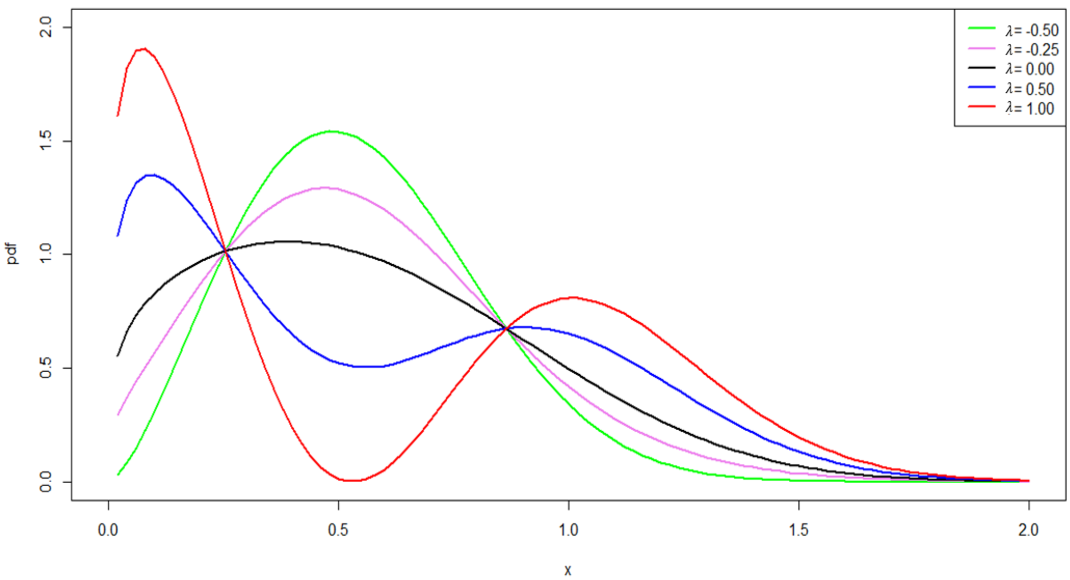

Figure 4. Graph of the pdf of the *TeSU-ExW* distribution for $\tau = 1.4$, $\beta = 2.5$, $\rho = 0.5$ and for some values of λ

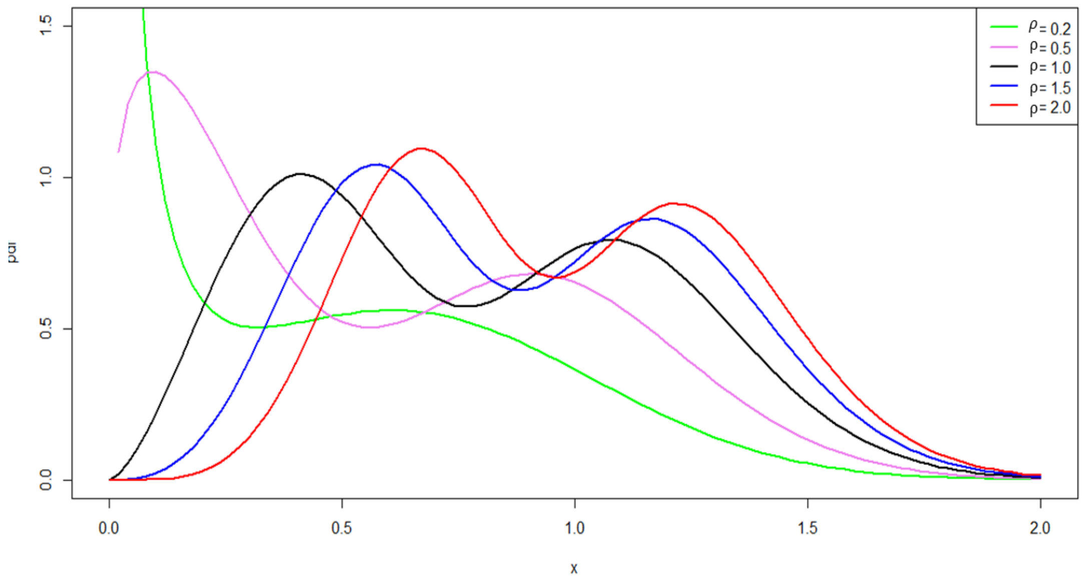

Figure 5. Graph of the pdf of the *TeSU-ExW* distribution for $\tau = 1.4$, $\beta = 2.5$, $\lambda = 0.5$ and for some values of ρ

4 Statistical Properties of eSU

The following lemmas and propositions discuss the asymptotic behavior of the eSU distribution as well as its raw moment, mean and variance.

Lemma 4.1. *Let T be a random variable with pdf given in equation (6). Then*

$$\lim_{t \to 0} f_{eSU}(t) = \lim_{t \to 1} f_{eSU}(t) = 1 + 2\lambda,$$

where $\lambda \in [-\frac{1}{2}, 1]$ and $t \in [0, 1]$.

Lemma 4.2. *Let T be a random variable with pdf given in equation (6). If $\lambda \in [-\frac{1}{2}, 0)$, then the mode of the eSU distribution is $\frac{1}{2}$.*

Proof. Taking the first derivative of the equation (6), we have

$$\begin{aligned}\frac{df_{eSU}(t)}{dt} &= \frac{d}{dt}\left[1 - \lambda + 3\lambda(2t-1)^2\right]\\ &= 6\lambda(2t-1)(2)\\ &= 12\lambda(2t-1).\end{aligned}$$

Equating this to zero to derive the maximum value of the distribution, we have $t = \dfrac{1}{2}$. Observe that $f''_{eSU}(t) = 24\lambda < 0$ if $\lambda \in [-\frac{1}{2}, 0)$, which corresponds to the maximum value of the function. $\square$

The next lemma shows some properties of the *pdf* of the eSU distribution for varied interval values of λ.

Lemma 4.3. *Let T be a random variable with pdf given in equation (6). Then the following statements holds:*

i. If $\lambda = 0$, then $f_{eSU}(t)$ has a constant function;

ii. If $\lambda \in [-\frac{1}{2}, 0)$, then $f_{eSU}(t)$ is increasing at $t \in [0, \frac{1}{2})$ and decreasing at $t \in (\frac{1}{2}, 0]$;

iii. If $\lambda \in (0, 1]$, then $f_{eSU}(t)$ is increasing at $t \in (\frac{1}{2}), 1]$ and decreasing at $t \in [0, \frac{1}{2})$;

iv. If $\lambda \in [-\frac{1}{2}, 0)$, then $f_{eSU}(t)$ is concave downward (inverted bathtub); and

v. If $\lambda \in (0, 1]$, then $f_{eSU}(t)$ is concave upward (bathtub shaped).

Proposition 4.4. *Let T be a random variable that follows an extended Standard U-quadratic (eSU) distribution. If $\lambda \in \left[-\frac{1}{2}, 1\right]$, then the rth moment of T is given by*

$$E\left[T^r\right] = \frac{(1+2\lambda)r^2 + (5-2\lambda)r + 6}{(r+1)(r+2)(r+3)}. \tag{16}$$

Proof. Recalling and expanding equation (6), we have $f_{eSU}(t) = 1 + 2\lambda - 12\lambda t + 12\lambda t^2$. For $t \in [0, 1]$, we have

$$\begin{aligned}E\left[T^r\right] &= \int_0^1 t^r f_{eSU}(t)dt\\ &= \int_0^1 t^r\left(1 + 2\lambda - 12\lambda t + 12\lambda t^2\right)d(t)\\ &= \frac{1+2\lambda}{r+1} - \frac{12\lambda}{r+2} + \frac{12\lambda}{r+3}\\ &= \frac{r^2 + 2\lambda r^2 + 5r - 2\lambda r + 6}{(r+1)(r+2)(r+3)}\\ &= \frac{(1+2\lambda)r^2 + (5-2\lambda)r + 6}{(r+1)(r+2)(r+3)}.\end{aligned}$$

$\square$

Proposition 4.5. *Let T be a random variable with moment given in equation (16). For $\lambda \in \left[-\frac{1}{2}, 1\right]$, the mean of T is $E[T] = \dfrac{1}{2}$ and the variance of T is $Var(T) = \dfrac{5+4\lambda}{60}$.*

Proof. The mean is easily obtained by letting $r = 1$ in (16), so that $E[T] = \dfrac{1}{2}$. To solve for $Var(T)$, we use the relation that $Var(T) = E[T^2] - (E[T])^2$. Substituting $r = 2$ in (16), we have $E[T^2] = \dfrac{5+\lambda}{15}$ so that

$$\begin{aligned} Var(T) &= \frac{5+\lambda}{15} - \left(\frac{1}{2}\right)^2 \\ &= \frac{5+4\lambda}{60}. \end{aligned}$$

□

5 Statistical Properties of $TeSU$-ExW

Lemma 5.1. *Let X be a random variable that follows $TeSU$-ExW. Then the cdf in (14) and pdf in (15) can be written as*

$$F_{TeSU\text{-}ExW}(x) = \sum_{j=0}^{\infty} (-1)^j \left[(1+2\lambda)\binom{\rho}{j} - 6\lambda\binom{2\rho}{j} + 4\lambda\binom{3\rho}{j}\right] e^{-j\tau x^{\beta}}$$

and

$$f_{TeSU\text{-}ExW}(x) = \tau\beta x^{\beta-1} \sum_{j=0}^{\infty} (-1)^j j \left[6\lambda\binom{2\rho}{j} - (1+2\lambda)\binom{\rho}{j} - 4\lambda\binom{3\rho}{j}\right] e^{-j\tau x^{\beta}}, \tag{17}$$

where $x > 0$, $\tau > 0$, $\beta > 0$, $\rho > 0$ and $\lambda \in [-\frac{1}{2}, 1]$.

Proof. Let X be a random variable that follows $TeSU$-ExW with *cdf* given by

$$F_{TeSU\text{-}ExW}(x) = (1+2\lambda)\left(1 - e^{-\tau x^{\beta}}\right)^{\rho} - 6\lambda\left(1 - e^{-\tau x^{\beta}}\right)^{2\rho} + 4\lambda\left(1 - e^{-\tau x^{\beta}}\right)^{3\rho}, \tag{18}$$

where $x > 0$, $\tau > 0$, $\beta > 0$, $\rho > 0$ and $\lambda \in [-\frac{1}{2}, 1]$.
Using the binomial generalized expansion for $\left(1 - e^{-\tau x^{\beta}}\right)^{\rho}$, $\left(1 - e^{-\tau x^{\beta}}\right)^{2\rho}$ and $\left(1 - e^{-\tau x^{\beta}}\right)^{3\rho}$, we have the following:

$$\left(1 - e^{-\tau x^{\beta}}\right)^{\rho} = \sum_{j=1}^{\infty} (-1)^j \binom{\rho}{j} e^{-j\tau x^{\beta}}; \tag{19}$$

$$\left(1 - e^{-\tau x^{\beta}}\right)^{2\rho} = \sum_{j=1}^{\infty} (-1)^j \binom{2\rho}{j} e^{-j\tau x^{\beta}}; \text{and} \tag{20}$$

$$\left(1 - e^{-\tau x^{\beta}}\right)^{3\rho} = \sum_{j=1}^{\infty} (-1)^j \binom{3\rho}{j} e^{-j\tau x^{\beta}}. \tag{21}$$

Inserting equations (19), (20) and (21) into (18) then we have

$$F_{TeSU\text{-}ExW}(x) = (1+2\lambda)\sum_{j=1}^{\infty} (-1)^j \binom{\rho}{j} e^{-j\tau x^{\beta}} - 6\lambda\sum_{j=1}^{\infty} (-1)^j \binom{2\rho}{j} e^{-j\tau x^{\beta}} + 4\lambda\sum_{j=1}^{\infty} (-1)^j \binom{3\rho}{j} e^{-j\tau x^{\beta}}.$$

Hence, it follows that

$$F_{TeSU\text{-}ExW}(x) = \sum_{j=0}^{\infty} (-1)^j \left[(1+2\lambda)\binom{\rho}{j} - 6\lambda\binom{2\rho}{j} + 4\lambda\binom{3\rho}{j} \right] e^{-j\tau x^{\beta}}. \tag{22}$$

Taking the derivative of equation (22), we get

$$f_{TeSU-ExW}(x) = \tau\beta x^{\beta-1} \sum_{j=0}^{\infty} (-1)^j j \left[6\lambda\binom{2\rho}{j} - (1+2\lambda)\binom{\rho}{j} - 4\lambda\binom{3\rho}{j} \right] e^{-j\tau x^{\beta}}.$$

□

Proposition 5.2. *Let X be a random variable that follows TeSU-ExW. Then the rth moment of X denoted by $\mathbb{E}[X^r]$ is given by*

$$\mathbb{E}\left[X^r\right] = \frac{1}{\tau^{\frac{r}{\beta}}}\Gamma\left(\frac{r}{\beta}+1\right) \sum_{j=0}^{\infty} \frac{(-1)^j \left[6\lambda\binom{2\rho}{j} - (1+2\lambda)\binom{\rho}{j} - 4\lambda\binom{3\rho}{j}\right]}{j^{\frac{r}{\beta}}}. \tag{23}$$

Proof. The rth raw moment is defined by $\mathbb{E}[X^r] = \int_0^\infty x^r f(x)dx$. Hence, using the *pdf* given in equation (17), we have

$$\begin{aligned}
\mathbb{E}[X^r] &= \int_0^\infty x^r \tau\beta x^{\beta-1} \sum_{j=0}^{\infty} (-1)^j j \left[6\lambda\binom{2\rho}{j} - (1+2\lambda)\binom{\rho}{j} - 4\lambda\binom{3\rho}{j} \right] e^{-j\tau x^{\beta}} dx \\
&= \tau\beta \sum_{j=0}^{\infty} (-1)^j j \left[6\lambda\binom{2\rho}{j} - (1+2\lambda)\binom{\rho}{j} - 4\lambda\binom{3\rho}{j} \right] \int_0^\infty x^{r+\beta-1} e^{-j\tau x^{\beta}} dx \\
&= \frac{1}{\tau^{\frac{r}{\beta}}}\Gamma\left(\frac{r}{\beta}+1\right) \sum_{j=0}^{\infty} \frac{(-1)^j \left[6\lambda\binom{2\rho}{j} - (1+2\lambda)\binom{\rho}{j} - 4\lambda\binom{3\rho}{j}\right]}{j^{\frac{r}{\beta}}}.
\end{aligned}$$

□

Proposition 5.3. *Let X be a random variable with moment given in equation (23). Then the mean and variance of X are, respectively,*

$$\mathbb{E}\left[X\right] = \frac{1}{\tau^{\frac{1}{\beta}}}\Gamma\left(\frac{1}{\beta}+1\right) \sum_{j=0}^{\infty} \frac{(-1)^j S_{j,\rho,\lambda}}{j^{\frac{1}{\beta}}}$$

and

$$Var(X) = \frac{1}{\tau^{\frac{2}{\beta}}} \left\{ \Gamma\left(\frac{2}{\beta}+1\right) \sum_{j=0}^{\infty} \frac{(-1)^j S_{j,\rho,\lambda}}{j^{\frac{2}{\beta}}} - \Gamma^2\left(\frac{1}{\beta}+1\right) \left[\sum_{j=0}^{\infty} \frac{(-1)^j S_{j,\rho,\lambda}}{j^{\frac{1}{\beta}}} \right]^2 \right\},$$

where $S_{j,\rho,\lambda} = \left[6\lambda\binom{2\rho}{j} - (1+2\lambda)\binom{\rho}{j} - 4\lambda\binom{3\rho}{j}\right]$.

Proof. The mean of X is obtained when $r = 1$ in (23). It follows that

$$\mathbb{E}\left[X\right] = \frac{1}{\tau^{\frac{1}{\beta}}}\Gamma\left(\frac{1}{\beta}+1\right) \sum_{j=0}^{\infty} \frac{(-1)^j S_{j,\rho,\lambda}}{j^{\frac{1}{\beta}}},$$

where $S_{j,\rho,\lambda} = \left[6\lambda\binom{2\rho}{j} - (1+2\lambda)\binom{\rho}{j} - 4\lambda\binom{3\rho}{j}\right]$. Now, $\mathbb{E}[X^2]$ is derived using $r = 2$ in equation (23). Thus,

$$\mathbb{E}\left[X^2\right] = \frac{1}{\tau^{\frac{2}{\beta}}}\Gamma\left(\frac{2}{\beta}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j S_{j,\rho,\lambda}}{j^{\frac{2}{\beta}}}.$$

It follows that the variance of X is given by

$$\begin{aligned}
Var[X] &= \mathbb{E}[X^2] - [\mathbb{E}[X]]^2 \\
&= \frac{1}{\tau^{\frac{2}{\beta}}}\Gamma\left(\frac{2}{\beta}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j S_{j,\rho,\lambda}}{j^{\frac{2}{\beta}}} - \left[\frac{1}{\tau^{\frac{1}{\beta}}}\Gamma\left(\frac{1}{\beta}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j S_{j,\rho,\lambda}}{j^{\frac{1}{\beta}}}\right]^2 \\
&= \frac{1}{\tau^{\frac{2}{\beta}}}\left\{\Gamma\left(\frac{2}{\beta}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j S_{j,\rho,\lambda}}{j^{\frac{2}{\beta}}} - \Gamma^2\left(\frac{1}{\beta}+1\right)\left[\sum_{j=0}^{\infty}\frac{(-1)^j S_{j,\rho,\lambda}}{j^{\frac{1}{\beta}}}\right]^2\right\}.
\end{aligned}$$

$\square$

Proposition 5.4. *Let X be a random variable with moment given in equation (23). Then the moment generating function (mgf) of X is given by*

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^r(-1)^j\left[6\lambda\binom{2\rho}{j} - (1+2\lambda)\binom{\rho}{j} - 4\lambda\binom{3\rho}{j}\right]}{r!(\tau j)^{\frac{r}{\beta}}}\Gamma\left(\frac{r}{\beta}+1\right).$$

Proof. By definition of mgf and using equation (23), we have

$$\begin{aligned}
M_X(t) &= \mathbb{E}(e^{tX}) \\
&= \int_0^{\infty} e^{tx} f_{TeSU-ExW}(x)dx.
\end{aligned}$$

Recall that $e^{tX} = \sum_{r=0}^{\infty}\frac{t^r}{r!}x^r$. Hence, we have

$$\begin{aligned}
M_X(t) &= \int_0^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r f_{TeSU-ExW}(x)dx \\
&= \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f_{TeSU-ExW}(x)dx \\
&= \sum_{r=0}^{\infty}\frac{t^r}{r!}\mathbb{E}[X^r].
\end{aligned}$$

Thus,

$$\begin{aligned}
M_X(t) &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{1}{\tau^{\frac{r}{\beta}}}\Gamma\left(\frac{r}{\beta}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\left[6\lambda\binom{2\rho}{j} - (1+2\lambda)\binom{\rho}{j} - 4\lambda\binom{3\rho}{j}\right]}{j^{\frac{r}{\beta}}} \\
&= \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^r(-1)^j\left[6\lambda\binom{2\rho}{j} - (1+2\lambda)\binom{\rho}{j} - 4\lambda\binom{3\rho}{j}\right]}{r!(\tau j)^{\frac{r}{\beta}}}\Gamma\left(\frac{r}{\beta}+1\right).
\end{aligned}$$

$\square$

Proposition 5.5. *Let X be a random variable that follows TeSU-ExW. Then the median of X is given by*

$$x_{med(TeSU-ExW)} = \begin{cases} \left(-\frac{1}{\tau} log\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{\beta}}, & if\ \lambda = 0 \\ \left(-\frac{1}{\tau} log\left(1-\left(\frac{1}{2}+2\sqrt{\frac{1}{2}\left(1-\frac{1}{\lambda}\right)}\ cos\left(\frac{90-2\pi}{3}\right)\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{\beta}}, & otherwise \end{cases}.$$

Proof. Consider the Structured Set of Skew–Kurtotic Transmutations proposed by Shaw [20], that is, for parameters α_1, α_2, we shall consider the polynomial family given by

$$P(z, \alpha_1, \alpha_2) = z - z(1-z)\left[\alpha_1 + \alpha_2\left(z - \frac{1}{2}\right)\right],$$

where $z \in [0,1]$ and that the non-negativity of the *pdf* P at the end points should satisfy

$$-1 - \frac{\alpha_2}{2} \leq \alpha_1 \leq 1 + \frac{\alpha_2}{2}.$$

Let u follows a uniform distribution $(0,1)$. Then the solution for the equation $P(z, \alpha_1, \alpha_2) = u$ is as follows:

$$z = \begin{cases} u, & if \alpha_1 = \alpha_2 = 0 \\ \dfrac{\alpha_1 - 1 + \sqrt{1 + \alpha_1(\alpha_1 + 4u - 2)}}{2\alpha_1}, & if \alpha_2 = 0 \\ \sqrt[3]{u}, & if \alpha_1 = \frac{3}{2}, \alpha_2 = 1 \\ 1 - \sqrt[3]{1-u}, & if \alpha_1 = -\frac{3}{2}, \alpha_2 = 1 \\ C(u, \alpha_1, \alpha_2), & otherwise \end{cases}$$

where $C(\cdot)$ is a function that denotes the general cubic (GC) solver for other cases. This function is processed by the following algorithm.

Step 1. Compute

$$Q = \frac{4\alpha_1^2 + 3(\alpha_2 - 4)\alpha_2}{36\alpha_2^2},$$

$$R = \frac{4\alpha_1^3 - 9\alpha_2\alpha_1(\alpha_2 + 2) + 27(1-2u)\alpha_2^2}{108\alpha_2^3}.$$

Step 2. If $R^2 > Q^3$, the equation has one real and two complex roots. In this case we have,

$$A = -sign(R)\left(|R| + \sqrt{R^2 - Q^3}\right)^{\frac{1}{3}};$$

$$B = \begin{cases} A, & if A = 0 \\ \dfrac{Q}{A}, & otherwise \end{cases};$$

$$C(u, \alpha_1, \alpha_2) = A + B - \frac{1}{3}\left(\frac{\alpha_1}{\alpha_2} - \frac{3}{2}\right).$$

Otherwise, the cubic has three real roots and this is done by setting

$$\theta = arccos\left(\frac{R}{\sqrt[3]{Q}}\right);$$

$$C(u,\alpha_1,\alpha_2) = -2\sqrt{Q}\ cos\left(\frac{\theta-2\pi}{3}\right) - \frac{1}{3}\left(\frac{\alpha_1}{\alpha_2} - \frac{3}{2}\right).$$

Observed that the *cdf* in (9) of the *TeSU-ExG* family can be rewritten as

$$F(x) = z - z(1-z)\left[\alpha_1 + \alpha_2\left(z - \frac{1}{2}\right)\right],$$

where $z = (B(x))^\rho$, $\alpha_1 = 0$ and $\alpha_2 = 4\lambda$, $\lambda \in [-0.5, 1]$. The inverse of $F(x)$ is a solution to the following equation

$$z = C(u,\alpha_1,\alpha_2) = (B(x))^\rho = C(u, 0, 4\lambda).$$

Hence, the given algorithm can be modified as follows:
Let u follows a uniform distribution $(0,1)$. Then,

Step 1*. Compute

$$Q = \frac{1}{2}\left(1-\frac{1}{\lambda}\right), \lambda \neq 0;$$

$$R = \frac{1-2u}{16\lambda},$$

Step 2*. If $R^2 > Q^3$ then

$$A = -sign(R)\left(|R| + \sqrt{R^2-Q^3}\right)^{\frac{1}{3}};$$

$$B = \begin{cases} A, & if\, A = 0 \\ \dfrac{Q}{A}, & otherwise \end{cases};$$

$$x = B^{-1}\left[\left(A+B+\frac{1}{2}\right)^{\frac{1}{\rho}}\right].$$

Otherwise,

$$\theta = arccos\left(\frac{R}{\sqrt{Q^3}}\right);$$

$$x = B^{-1}\left[\left(\frac{1}{2} - 2\sqrt{Q}\ cos\left(\frac{\theta-2\pi}{3}\right)\right)^{\frac{1}{\rho}}\right],$$

where $B^{-1}(x)$ is the inverse function of any baseline distribution function $B(x)$. If $\lambda = 0$, then $x = B^{-1}\left(u^{\frac{1}{\rho}}\right)$. The updated algorithm can be used for generating random numbers that follows any *TeSU-ExG* distribution. Consequently, the median of *TeSU-ExG* family can be computed by taking $u = \frac{1}{2}$. Since $R \leq Q$ for $u = \frac{1}{2}$, then the median is given by

$$x_{med} = \begin{cases} B^{-1}\left[\left(\frac{1}{2}\right)^{\frac{1}{\rho}}\right], & if\ \lambda = 0 \\ B^{-1}\left\{\left[\frac{1}{2} - 2\sqrt{\frac{1}{12}\left(1-\frac{1}{\lambda}\right)}\ cos\left(\frac{90-2\pi}{3}\right)\right]^{\frac{1}{\rho}}\right\}, & otherwise \end{cases}. \tag{24}$$

Setting $B(x)$ to be the *cdf* in equation (10) of the Weibull distribution. Then, the median of the *TeSU-ExW* is given by

$$x_{med(TeSU\text{-}ExW)} = \begin{cases} \left(-\frac{1}{\tau}log\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{\beta}}, & if\ \lambda = 0 \\ \left(-\frac{1}{\tau}log\left(1-\left(\frac{1}{2}+2\sqrt{\frac{1}{2}\left(1-\frac{1}{\lambda}\right)}\ cos\left(\frac{90-2\pi}{3}\right)\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{\beta}}, & otherwise \end{cases}.$$

$\square$

Proposition 5.6. *Let X be a random variable with pdf given in equation (15). Then the critical values are the roots of*

$$\frac{12\lambda\rho\beta\tau x^{\beta}e^{-\tau x^{\beta}}(1-e^{-\tau x^{\beta}})^{\rho}(2(1-e^{-\tau x^{\beta}})-1)}{1-\lambda+3\lambda(2(1-e^{-\tau x^{\beta}})^{\rho}-1)^{2}} = 1-\beta(1-\tau x^{\beta})-(1-\beta+\rho\beta\tau x^{\beta})e^{-\tau x^{\beta}}.$$

Proof. Consider the *pdf* given in equation (15). Taking the log of pdf (15), we have

$$\begin{aligned}\log(f_{TeSU\text{-}ExW}(x)) &= \log\left\{\beta\rho\tau x^{\beta-1}e^{-\tau x^{\beta}}\left(1-e^{-\tau x^{\beta}}\right)^{\rho-1}\left\{1-\lambda+3\lambda\left[2(1-e^{-\tau x^{\beta}})^{\rho}-1\right]^{2}\right\}\right\}\\ &= \log(\rho)+\log(\tau)+\log(\beta)+(\beta-1)\log(x)-\tau x^{\beta}+(\rho-1)\log(1-e^{-\tau x^{\beta}})\\ &\quad+\log\left\{1-\lambda+3\lambda\left[2(1-e^{-\tau x^{\beta}})^{\rho}-1\right]^{2}\right\}.\end{aligned}$$

Next, taking the derivative of the $\log(f_{TeSU-ExW}(x))$, we have

$$\begin{aligned}\frac{d}{dx}\left(f_{TeSU\text{-}ExW}(x)\right) &= \frac{\beta-1}{x}-\beta\tau x^{\beta-1}+\frac{(\rho-1)\beta\tau x^{\beta-1}e^{-\tau x^{\beta}}}{1-e^{-\tau x^{\beta}}}\\ &\quad+\frac{12\lambda\rho\beta\tau x^{\beta}e^{-\tau x^{\beta}}\left(1-e^{-\tau x^{\beta}}\right)^{\rho-1}\left(2\left(1-e^{-\tau x^{\beta}}\right)-1\right)}{x\left\{1-\lambda+3\lambda(2\left(1-e^{-\tau x^{\beta}}\right)^{\rho}-1)^{2}\right\}}.\end{aligned}$$

Moreover, equating $\frac{d}{dx}\left(f_{TeSU\text{-}ExW}(x)\right)=0$, we have

$$\begin{aligned}0 &= \frac{\beta-1}{x}-\beta\tau x^{\beta-1}+\frac{(\rho-1)\beta\tau x^{\beta-1}e^{-\tau x^{\beta}}}{1-e^{-\tau x^{\beta}}}\\ &\quad+\frac{12\lambda\rho\beta\tau x^{\beta}e^{-\tau x^{\beta}}\left(1-e^{-\tau x^{\beta}}\right)^{\rho-1}\left(2\left(1-e^{-\tau x^{\beta}}\right)-1\right)}{x\left\{1-\lambda+3\lambda(2\left(1-e^{-\tau x^{\beta}}\right)^{\rho}-1)^{2}\right\}}.\end{aligned}$$

Thus,

$$\frac{12\lambda\rho\beta\tau x^{\beta}e^{-\tau x^{\beta}}(1-e^{-\tau x^{\beta}})^{\rho}(2(1-e^{-\tau x^{\beta}})-1)}{1-\lambda+3\lambda(2(1-e^{-\tau x^{\beta}})^{\rho}-1)^{2}} = 1-\beta(1-\tau x^{\beta})-(1-\beta+\rho\beta\tau x^{\beta})e^{-\tau x^{\beta}}.$$

$\square$

Remark 5.7. If the point $x_0 = x$ is the root of equation (5.6), then we can classify it as local maximum, local minimum or inflection point when we have, respectively, $\delta(x_0) < 0$, $\delta(x_0) > 0$ and $\delta(x_0) = 0$, where $\delta(x)$ is the second derivative of the *pdf* in (15). x_0 is a mode(s) of *TeSU-ExW* if $\delta(x_0) < 0$.

6 Conclusion and Recommendations

This paper derives an extension of the Standard U-quadratic distribution called as extended Standard U-quadratic distribution (eSU) and explore some of its properties like its limiting behavior, moment, mean, variance and mode. Moreover, a new G family of distributions called as the T-extended Standard U-quadratic-G family of distributions ($TeSU$-G) is introduced and its extension labeled as $TeSU$-Exponentiated G family of distributions ($TeSU$-ExG) is also presented. A special model of the family called $TeSU$-Weibull ($TeSU$-W) distribution and its extension labeled as $TeSU$-Exponentiated Weibull distribution ($TeSU$-ExW) are also derived. In addition, some properties of the $TeSU$-ExW like its moment, moment generating function, mean, variance, median and mode are computed. Furthermore, this paper focus only on the Weibull distribution as a baseline of the proposed family of distributions. So, for future studies in this field, it is recommended to use any other distributions like normal, log-normal, Dagum and Lomax distributions as a baseline for the proposed family of distributions and explore the possibility of its application to lifetime data analysis.

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