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θ^ω-Open Set and its Corresponding Topological Concepts

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Abstract

In this paper, we introduce a different version of open set called θ^ω-open set, and then described its connection to the other well-known concepts such as the classical open, θ-open, and ω-open sets. It is worth noting that the family of all θ^ω-open sets forms a topology. We also define and investigate the concepts of θ^ω-interior and θ^ω-closure of a set. The concepts of strongly θ^ω-open function, strongly θ^ω-closed function, θ^ω-open function, and θ^ω-closed function are defined and characterized. Finally, related concepts such as θ^ω-continuous function, strongly θ^ω-continuous function, and θ^ω-connected are investigated further.

1 Introduction and Preliminaries

Several mathematicians are still drawn to proposing alternative topological concepts that can replace concepts with stronger or weaker properties. This is due to the work done by Levine [15] in 1963 where he introduced the concepts of semi-open, semi-closed set, and semi-continuity of a function. This then generated new results, some of which are generalization of existing ones.

In 1968, Velicko [17] introduced the concepts of θ-closure and θ-interior of a subset of a topological space and subsequently defined the concepts of θ-continuity of a function in topological spaces. Several authors then have obtained results related to θ-open sets, see [1, 4, 5, 6, 7, 8].

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The θ-closure and θ-interior of A are, respectively, denoted and defined by

$$Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \varnothing \text{ for every open set } U \text{ containing } x\}$$

and

$$Int_\theta(A) = \{x \in X : Cl(U) \subseteq A \text{ for every open set } U \text{ containing } x\},$$

where $Cl(U)$ is the closure of U in X. A subset A of X is θ-closed if $Cl_\theta(A) = A$ and θ-open if $Int_\theta(A) = A$. Equivalently, A is θ-open if and only if $X \setminus A$ is θ-closed. It is known that the collection $\mathcal{T}_\theta$ of all θ-open sets forms a topology on X, which is strictly coarser that $\mathcal{T}$.

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In 1982, Hdeib [10] introduced the concepts of ω-open and ω-closed sets and ω-closed mappings on a topological space. Several papers have studied the concepts of ω-open sets and its corresponding topological concepts, such as [14, 16].

A point x of a topological space X is called a condensation point of $A \subseteq X$ if for each open set G containing x, $G \cap A$ is uncountable. A subset B of X is ω-closed if it contains all of its condensation points. The complement of B is ω-open. Equivalently, a subset U of X is ω-open if and only if for each $x \in U$, there exists an open set O containing x such that $O \setminus U$ is countable.

A topological space X is said to be connected (resp., θ-connected, ω-connected) if X cannot be written as the union of two nonempty disjoint open (resp., θ-open, ω-open) sets. Otherwise, X is disconnected (resp., θ-disconnected, ω-disconnected).

It is known that $Int_\theta(A)$ [12] (resp., $Int_\omega(A)$ [14]) is the largest θ-open (resp., ω-open) set contained in A and $Cl_\theta(A)$ [12] (resp., $Cl_\omega(A)$ [14]) is the smallest θ-closed (resp., ω-closed) set containing A. Moreover, $x \in Int_\theta(A)$ [17] (resp., $x \in Int_\omega(A)$ [14]) if and only if there exists an open (resp., ω-open) set U containing x such that $Cl(U) \subseteq A$ (resp., $U \subseteq A$) and $x \in Cl_\theta(A)$ [17] (resp., $x \in Cl_\omega(A)$ [14]) if and only if for every open (resp., ω-open) set U containing x, $Cl(U) \cap A \neq \varnothing$ (resp., $U \cap A \neq \varnothing$). It is worth noting that $Int_\theta(A) \subseteq Int(A)$ [12] (resp., $Int_\theta(A) \subseteq Int_\omega(A)$ [14]) and $Cl(A) \subseteq Cl_\theta(A)$ [12] (resp., $Cl_\omega(A) \subseteq Cl_\theta(A)$ [14]), as well as $Cl_\omega(A) \subseteq Cl(A)$ [2], for any subset A of a topological space X.

Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\mathcal{T}_\alpha$ be the topology on Y_α. The Tychonoff topology on $\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $\langle U_\alpha \rangle = \mathrm{p}_\alpha^{-1}(U_\alpha)$, where $p_\alpha : \prod\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$, the αth coordinate projection map is defined by $p_\alpha(\langle Y_\beta \rangle) = y_\alpha$, U_α ranges over all members of $\mathcal{T}_\alpha$, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \cdots, \alpha_n$ and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}, U_{\alpha_2} \subseteq Y_{\alpha_2}, \cdots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_\alpha^{-1}(U_{\alpha_1}) \cap p_\alpha^{-1}(U_{\alpha_2}) \cap \cdots \cap p_\alpha^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, \cdots, B_{\alpha_k} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, \cdots, k\}$.

Now, the projection map $p_\alpha : \prod\{Y_\alpha : \alpha \in A\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in A$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in A\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α-th coordinate projection map.

In this paper, we introduced a new class of open set called θ^ω-open set. Related concepts such as θ^ω-open (resp., closed) and strongly θ^ω-open (resp., strongly θ^ω-closed) functions, θ^ω-continuous, and θ^ω-connectedness are defined and characterized.

2 θ^ω-Open and θ^ω-Closed Functions

In this section, we define and characterize the concepts of θ^ω-open (resp., θ^ω-closed) and strongly θ^ω-open (resp., strongly θ^ω-closed) functions. Throughout, if no confusion arises, let X and Y be topological spaces.

Definition 2.1. Let X be a topological space. A subset A of X is said to be θ^ω-*open* if for every $x \in A$, there exists an ω-open set U containing x such that $Cl_\omega(U) \subseteq A$. A subset B of X is said to be θ^ω-*closed* if its complement $X \setminus B$ is θ^ω-open.

Theorem 2.2. *Let X be a topological space and $A \subseteq X$. Then the following holds:*

(i) *If A is θ-open, then A is θ^ω-open; and*

(ii) *If A is θ^ω-open, then A is ω-open.*

Proof. (i) Suppose that A is θ-open in X and let $x \in A$. Then there exists an open set U with $x \in U$ such that $Cl(U) \subseteq A$. Since U is open, U is also ω-open. Also, $Cl_\omega(U) \subseteq Cl(U) \subseteq A$. Hence, $Cl_\omega(U) \subseteq A$. Therefore, A is θ^ω-open.

(ii) Suppose that A is θ^ω-open. Then for every $x \in A$, there exists an ω-open set U containing x such that $U \subseteq Cl_\omega(U) \subseteq A$. Since U is ω-open, there exists an open set V containing x such that $V \setminus U$ is countable. Note that $V \setminus A \subseteq V \setminus U$. Since $V \setminus U$ is countable, $V \setminus A$ is also countable. Hence, A is ω-open. □

Corollary 2.3. *Let X be a topological space and $A \subseteq X$. Then the following holds:*

(i) *If A is θ-closed, then A is θ^ω-closed; and*

(ii) *If A is θ^ω-closed, then A is ω-closed.*

Proof. (i) Suppose A is θ-closed, then $X \setminus A$ is θ-open. Thus, by Theorem 2.2 (i), $X \setminus A$ is θ^ω-open. Therefore, $X \setminus (X \setminus A) = A$ is θ^ω-closed.

(ii) Suppose that A is θ^ω-closed, then by Definition 2.1, $X \setminus A$ is θ^ω-open. Thus, by Theorem 2.2 (ii), $X \setminus A$ is ω-open. Therefore, $X \setminus (X \setminus A) = A$ is ω-closed. □

The proof of the following Lemma is omitted since it is an immediate consequence of Definition 2.1.

Lemma 2.4. *Let X be a topological space and $A \subseteq X$. If A is both ω-open and ω-closed, then A is θ^ω-open.*

Remark 2.5. θ^ω-open (resp., θ^ω-closed) sets and open (resp., closed) sets are two independent notions.

Example 2.6. Consider $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}\}$.

Consider the set $\{b, c\}$, which is not an open set. Since X is countable, every subset of X is ω-open and ω-closed. It follows that $\{b, c\}$ is ω-open and ω-closed. Hence, by Lemma 2.4, $\{b, c\}$ is θ^ω-open.

Example 2.7. Let $X = [\sqrt{2}, 2)$ be an interval in $\mathbb{R}$ with the topology

$$\mathcal{T} = \{\varnothing, [\sqrt{2}, 2), \mathbb{Q}^c \cap [\sqrt{2}, 2)\}.$$

For any set $S \subseteq [\sqrt{2}, 2)$, let $S^c := [\sqrt{2}, 2) \setminus S$. We note first that if U is ω-open, then U must be uncountable since $[\sqrt{2}, 2)$ and $\mathbb{Q}^c \cap [\sqrt{2}, 2)$ are uncountable. Let $A = \mathbb{Q}^c \cap [\sqrt{2}, 2)$ which is an open set. By [8, Remark 4], A is ω-open. We will show that A is not θ^ω-open in X. Suppose on the contrary that A is θ^ω-open. Then, for every $x \in A$, there exists an ω-open set U containing x such that $U \subseteq Cl_\omega(U) \subseteq A$. Since U is ω-open, either $[\sqrt{2}, 2) \setminus U$ or $[\mathbb{Q}^c \cap [\sqrt{2}, 2)] \setminus U$ is countable. If $[\mathbb{Q}^c \cap [\sqrt{2}, 2)] \setminus U$ is countable, then

$$[\sqrt{2}, 2) \setminus U = [[\mathbb{Q}^c \cap [\sqrt{2}, 2)] \setminus U] \cup [\mathbb{Q} \cap [\sqrt{2}, 2)]$$

is countable. Either case, $[\sqrt{2}, 2) \setminus U$ is countable.

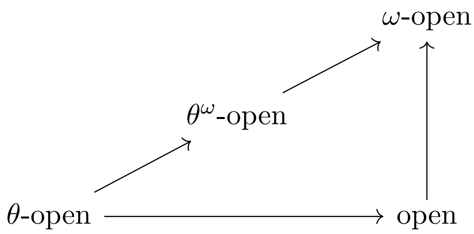
Next, we show that $A = \mathbb{Q}^c \cap [\sqrt{2}, 2)$ is not ω-closed. Suppose that A is ω-closed. Then $X \setminus A = \mathbb{Q} \cap [\sqrt{2}, 2)$ is ω-open, a contradiction since $X \setminus A$ is countable. Thus, A is not ω-closed.

It follows that $U \subseteq Cl_\omega(U) \subsetneq A$. This means that there exists $y \in A$ such that $y \notin Cl_\omega(U)$. Hence, by [14, Lemma 2 (iii)], there exists an ω-open set B containing y such that $B \cap U = \varnothing$. This implies that $B \subseteq [\sqrt{2}, 2) \setminus U$. This is a contradiction since B is uncountable and $[\sqrt{2}, 2) \setminus U$ is countable. Accordingly, A is not θ^ω-open.

Similar argument above can be used to verify that θ^ω-closed sets and closed sets are two independent notions.

In view of Theorem 2.2, Corollary 2.3, and Remark 2.5, we have the following remark.

Remark 2.8. The following diagram holds for a subset of topological space.



We remark that the above diagram also holds for its respective closed sets. The reverse implications for ω-open and θ^ω-open sets and θ^ω-open and θ-open sets are not true as shown in the following examples. Counterexamples for the other reverse implication are found in [8, p.295].

Example 2.9. Consider $X = \{a, b, c, d\}$ with a topology $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Consider $A = \{c, d\}$. Since X is countable, all subsets of X is ω-open and ω-closed. Hence, $\{c, d\}$ is ω-open and ω-closed. Thus, by Lemma 2.4, A is θ^ω-open. Clearly, A is not open. Thus, A is not θ-open (see [8]).

Example 2.10. Consider again Example 2.7 with the same topology. Let $A = \mathbb{Q}^c \cap [\sqrt{2}, 2)$. It is already shown in Example 2.7 that A is ω-open but not θ^ω-open.

Lemma 2.11. *Let X be a topological space and let $A, B \subseteq X$. Then $Cl_\omega(A \cap B) \subseteq Cl_\omega(A) \cap Cl_\omega(B)$.*

Proof. Let $A, B \subseteq X$. Since $A \subseteq Cl_\omega(A)$ and $B \subseteq Cl_\omega(B)$. Hence, we have $A \cap B \subseteq Cl_\omega(A) \cap Cl_\omega(B)$. Since $Cl_\omega(A \cap B)$ is the smallest ω-closed set containing $A \cap B$. Thus, we have $Cl_\omega(A \cap B) \subseteq Cl_\omega(A) \cap Cl_\omega(B)$. □

Theorem 2.12. *Let $\mathcal{T}_{\theta^\omega}$ be the family of θ^ω-open subsets of a topological space X. Then $\mathcal{T}_{\theta^\omega}$ forms a topology on X.*

Proof. Clearly, $X, \varnothing \in \mathcal{T}_{\theta^\omega}$.

Now, let $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of θ^ω-open subsets of X. Let $x \in \bigcup_{\alpha \in \mathcal{A}} A_\alpha$. Then $x \in A_{\alpha_0}$ for some $\alpha_0 \in \mathcal{A}$. Since A_{α_0} is θ^ω-open, then there exist an ω-open set U_{α_0} with $x \in U_{\alpha_0}$ such that $Cl_\omega(U_{\alpha_0}) \subseteq A_{\alpha_0} \subseteq \bigcup_{\alpha \in \mathcal{A}} A_\alpha$. Thus, $\bigcup_{\alpha \in \mathcal{A}} A_\alpha \in \mathcal{T}_{\theta^\omega}$.

Lastly, let A_1 and $A_2 \in \mathcal{T}_{\theta^\omega}$ and let $x \in A_1 \cap A_2$. Since $A_1 \cap A_2$ is θ^ω-open, then there exists an ω-open sets U_1 and U_2 with $x \in U_1 \cap U_2$ such that $Cl_\omega(U_1) \subseteq A_2$ and $Cl_\omega(U_2) \subseteq A_2$. Since $\mathcal{T}_\omega$ forms a topology, it follows that $U_1 \cap U_2$ is ω-open set containing x, then by Lemma 2.11, we have $Cl_\omega(U_1 \cap U_2) \subseteq Cl_\omega(U_1) \cap Cl_\omega(U_2) \subseteq A_1 \cap A_2$. Hence, $A_1 \cap A_2 \in \mathcal{T}_{\theta^\omega}$.

Therefore, $\mathcal{T}_{\theta^\omega}$ is a topology on X. □

Definition 2.13. Let X be a topological space and $A \subseteq X$.

(i) The θ^ω-interior of A denoted by $Int_{\theta^\omega}(A)$, is defined by $Int_{\theta^\omega}(A) = \bigcup\{U : U$ is a θ^ω-open set and $U \subseteq A\}$. We note that in view of Theorem 2.12, $Int_{\theta^\omega}(A)$ is the largest θ^ω-open set contained in A. Moreover, $x \in Int_{\theta^\omega}(A)$ if and only if there exists a θ^ω-open set U with $x \in U$ such that $U \subseteq A$.

(ii) The θ^ω-closure of A denoted by $Cl_{\theta^\omega}(A)$, is defined by $Cl_{\theta^\omega}(A) = \bigcap\{F : F$ is a θ^ω-closed set and $A \subseteq F\}$. Then by Theorem 2.12, $Cl_{\theta^\omega}(A)$ is the smallest θ^ω-closed set containing A. Moreover, $x \in Cl_{\theta^\omega}(A)$ if and only if for every θ^ω-open set U containing x, $U \cap A \neq \varnothing$.

Remark 2.14. Let X be a topological space and $A, B \subseteq X$. Then the following statements hold:

(i) $Int_{\theta^\omega}(A) \subseteq A$.

(ii) A is θ^ω-open if and only if $A = Int_{\theta^\omega}(A)$.

(iii) $A \subseteq B$ implies that $Int_{\theta^\omega}(A) \subseteq Int_{\theta^\omega}(B)$.

(iv) $Int_{\theta^\omega}(A) = Int_{\theta^\omega}(Int_{\theta^\omega}(A))$.

(v) $Int_{\theta^\omega}(A) \cap Int_{\theta^\omega}(B) = Int_{\theta^\omega}(A \cap B)$.

(vi) $A \subseteq Cl_{\theta^\omega}(A)$.

(vii) A is θ^ω-closed if and only if $A = Cl_{\theta^\omega}(A)$.

(viii) $A \subseteq B$ implies that $Cl_{\theta^\omega}(A) \subseteq Cl_{\theta^\omega}(B)$.

(ix) $Cl_{\theta^\omega}\ (Cl_{\theta^\omega}(A)) = Cl_{\theta^\omega}(A)$.

(x) $Cl_{\theta^\omega}(A) \cup Cl_{\theta^\omega}(B) = Cl_{\theta^\omega}(A \cup B)$.

(xi) $Int_{\theta^\omega}(X \setminus A) = X \setminus Cl_{\theta^\omega}(A)$.

(xii) $Cl_{\theta^\omega}(X \setminus A) = X \setminus Int_{\theta^\omega}(A)$.

(xiii) $x \in Int_{\theta^\omega}(A)$ if and only if there exists an ω-open set U containing x such that $Cl_\omega(U) \subseteq A$.

(xiv) $x \in Cl_{\theta^\omega}(A)$ if and only if for each ω-open set U containing x, $Cl_\omega(U) \cap A \neq \varnothing$.

(xv) $Int_\theta(A) \subseteq Int_{\theta^\omega}(A) \subseteq Int_\omega(A)$.

(xvi) $Cl_\omega(A) \subseteq Cl_{\theta^\omega}(A) \subseteq Cl_\theta(A)$.

In view of Theorem 2.2 and Remark 2.8 we have the following corollary.

Corollary 2.15. *Let* $(X, \mathcal{T})$ *be a topological space. Then* $\mathcal{T}_\theta \subseteq \mathcal{T}_{\theta^\omega} \subseteq \mathcal{T}_\omega$.

We shall give some characterizations of strongly θ^ω-open (resp., strongly θ^ω-closed) and θ^ω-open (resp., θ^ω-closed) functions.

Definition 2.16. Let X and Y be a topological spaces. A function $f : X \to Y$ is said to be

(i) *strongly θ^ω-open* (resp., strongly θ^ω-closed) on X if $f(G)$ is θ^ω-open (resp. θ^ω-closed) in Y for every open (resp., closed) set G in X.

(ii) *θ^ω-open* (resp., θ^ω-closed) on X if $f(G)$ is θ^ω-open (resp., θ^ω-closed) in Y for every θ-open (resp., θ-closed) set G in X.

In [8, Remark 4], if A is θ-open (resp., θ-closed), then A is open (resp., closed). Then we have the following remarks.

Remark 2.17. Every strongly θ^ω-open function is a θ^ω-open function but the converse is not necessarily true.

Example 2.18. Let $X = Y = \mathbb{R}$ with topologies $\mathcal{T}_X = \{\varnothing, \mathbb{R}, \mathbb{N}\}$ and $\mathcal{T}_Y = \{\varnothing, \mathbb{R}, \mathbb{R} \setminus \mathbb{N}\}$.

Consider $f : \mathbb{R} \to \mathbb{R}$ be the identity function, that is, $f(x) = x$ for all $x \in \mathbb{R}$. Observe that the only θ-open sets in X are $\varnothing$ and X. Also, $f(\varnothing) = \varnothing$ and $f(X) = Y$ are θ^ω-open in Y. Thus, f is θ^ω-open on X.

To show that f is not strongly θ^ω-open, we will verify first that $\mathbb{N}$ is not ω-open in Y. Suppose that $\mathbb{N}$ is ω-open in Y. Then for all $x \in \mathbb{N}$, there exists an open set V containing x such that $V \setminus \mathbb{N}$ is countable. Since $\mathbb{R}$ is the only open set containing $x \in \mathbb{N}$, $\mathbb{R} \setminus \mathbb{N}$ is uncountable, a contradiction. Hence, $\mathbb{N}$ is not ω-open. Accordingly, $\mathbb{N}$ is not θ^ω-open, by Theorem 2.2 (ii).

Next, note that $\mathbb{N}$ is not θ^ω-open in X, it follows that $f(\mathbb{N}) = \mathbb{N}$ is not θ^ω-open in Y. Thus, f is not strongly θ^ω-open on X.

Remark 2.19. Every strongly θ^ω-closed function is a θ^ω-closed function but the converse is not necessarily true.

Consider again Example 2.18 with the same topologies. Note that the only θ-closed in X are $\varnothing$ and $\mathbb{R}$. Also, $f(\varnothing) = \varnothing$ and $f(X) = Y$ are θ^ω-closed in Y. Thus, f is θ^ω-closed on X.

Next, we show that f is not strongly θ^ω-closed on X. Observe that the closed sets in X are $\varnothing$, $\mathbb{R}$, and $\mathbb{R} \setminus \mathbb{N}$. Also, $f(\mathbb{R} \setminus \mathbb{N}) = \mathbb{R} \setminus \mathbb{N}$. We have shown in Example 2.18, that $\mathbb{N}$ is not θ^ω-open in Y. Thus, $\mathbb{R} \setminus \mathbb{N}$ is not θ^ω-closed in Y. Therefore, f is not strongly θ^ω-closed on X.

Theorem 2.20. *Let X and Y be a topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is strongly θ^ω-open on X.*

(ii) *$f(Int(A)) \subseteq Int_{\theta^\omega}(f(A))$ for every $A \subseteq X$.*

(iii) *$f(B)$ is θ^ω-open for every basic open set B in X.*

(iv) *For each $p \in X$ and every open set O in X containing p, there exists an ω-open set W in Y containing $f(p)$ such that $Cl_\omega(W) \subseteq f(O)$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that f is strongly θ^ω-open on X. Let A $\subseteq X$. Note that $Int(A) \subseteq A$, this means that $f(Int(A)) \subseteq f(A)$. Since f is strongly θ^ω-open and $Int(A)$ is open in X, $f(Int(A))$ is θ^ω-open set contained in $f(A)$. Observe that $Int_{\theta^\omega}(f(A))$ is the largest θ^ω-open set contained in $f(A)$. Thus, $f(Int(A)) \subseteq Int_{\theta^\omega}(f(A))$.

(ii) $\Rightarrow$ (iii): Assume that (ii) holds and let B be a basic open set in X. Then by assumption, $f(B) = f(Int(B)) \subseteq Int_{\theta^\omega}(f(B)) \subseteq f(B)$. Hence, $Int_{\theta^\omega}(f(B)) = f(B)$. Thus, $f(B)$ is θ^ω-open in X.

(iii) $\Rightarrow$ (iv): Suppose that (iii) holds. Let $p \in X$ and let O be an open in X containing p. Since O is open, then there exists a basic open set B with $p \in B$ such that $B \subseteq O$. Thus,

$f(p) \in f(B) \subseteq f(O)$. Then $f(B)$ is θ^ω-open in Y, by assumption. Thus, there exists an ω-open set W in Y with $f(p) \in W$ such that $Cl_\omega(W) \subseteq f(B) \subseteq f(O)$.

(iv) $\Rightarrow$ (i): Assume that (iv) holds. Let O be open in X and let $y \in f(O)$. Then there exists p with $p \in O$ such that $f(p) = y$. By assumption, there exists an ω-open set W in Y with $y \in W$ such that $Cl_\omega(W) \subseteq f(O)$. Thus, $f(O)$ is θ^ω-open in Y. Therefore, f is strongly θ^ω-open on X. □

Theorem 2.21. *Let X and Y be a topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is strongly θ^ω-closed on X.*

(ii) *$Cl_{\theta^\omega}(f(A)) \subseteq f(Cl(A))$ for every $A \subseteq X$.*

Proof. (i) $\Rightarrow$ (ii): Assume that f is strongly θ^ω-closed on X and let $A \subseteq X$. Since $A \subseteq Cl(A)$, then $f(A) \subseteq f(Cl(A))$. Note that $Cl(A)$ is closed in X and $f(Cl(A))$ is θ^ω-closed in Y. Thus, $Cl_{\theta^\omega}(f(A)) \subseteq f(Cl(A))$ since $Cl_{\theta^\omega}(f(A))$ is the smallest θ^ω-closed set containing $f(A)$.

(ii) $\Rightarrow$ (i): Let F be closed in X. Thus, $F = Cl(F)$. This means that $f(F) = f(Cl(F))$. By assumption, we have $f(F) \subseteq Cl_{\theta^\omega}(f(F)) \subseteq f(Cl(F)) = f(F)$. Thus, $Cl_{\theta^\omega}(f(F)) = f(F)$, which means $f(F)$ is θ^ω-closed in Y. Therefore, f is strongly θ^ω-closed on X. □

Theorem 2.22. *Let X and Y be a topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ^ω-open on X.*

(ii) *$f(Int_\theta(A)) \subseteq Int_{\theta^\omega}(f(A))$ for every $A \subseteq X$.*

(iii) *For each $p \in X$ and every open set O in X containing p, there exists an ω-open set W in Y containing $f(p)$ such that $Cl_\omega(W) \subseteq f(Cl(O))$.*

Proof. (i) $\Rightarrow$ (ii): Suppose f is θ^ω-open on X. Let A $\subseteq X$. Since $Int_\theta(A) \subseteq A$, $f(Int_\theta(A)) \subseteq f(A)$. Now, note that f is θ^ω-open and $Int_\theta(A)$ is θ-open in X, so we have $f(Int_\theta(A))$ is a θ^ω-open set contained in $f(A)$. Since $Int_{\theta^\omega}(f(A))$ is the largest θ^ω-open set contained in $f(A)$, $f(Int_\theta(A)) \subseteq Int_{\theta^\omega}(f(A))$.

(ii) $\Rightarrow$ (iii): Assume that (ii) holds. Then let $p \in X$ and O be an open in X containing p. Then there exists an open set U containing p such that $U \subseteq O$. This means $x \in U \subseteq Cl(U) \subseteq Cl(O)$. It follows that $p \in Int_\theta(Cl(O))$. By assumption, $f(p) \in f(Int_\theta(Cl(O))) \subseteq Int_{\theta^\omega}(f(Cl(O)))$. By Remark 2.14 (xiv), there exists an ω-open set W in Y containing $f(p)$ such that $Cl_\omega(W) \subseteq f(Cl(O))$.

(iii) $\Rightarrow$ (i): Assume that (iii) holds. Let O be a θ-open in X and let $y \in f(O)$. Then, there exists p with $p \in O$ such that $f(p) = y$. Since O is θ-open, there exists an open set V containing p such that $Cl(V) \subseteq O$. By assumption, there exists an ω-open set W in Y with $y \in W$ such that $Cl_\omega(W) \subseteq f(Cl(V)) \subseteq f(O)$. Thus, $f(O)$ is θ^ω-open in Y. Therefore, f is θ^ω-open on X. □

Theorem 2.23. *Let X and Y be a topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ^ω-closed on X.*

(ii) *$Cl_{\theta^\omega}(f(A)) \subseteq f(Cl_\theta(A))$ for every $A \subseteq X$.*

Proof. (i) $\Rightarrow$ (ii): Assume that f is θ^ω-closed on X and let $A \subseteq X$. Since $A \subseteq Cl_\theta(A)$, then $f(A) \subseteq f(Cl_\theta(A))$. Since $Cl_\theta(A)$ is θ-closed in X, $f(Cl_\theta(A))$ is θ^ω-closed in Y. Since $Cl_{\theta^\omega}(f(A))$ is the smallest θ^ω-closed set containing $f(A)$. Thus, $Cl_{\theta^\omega}(f(A)) \subseteq f(Cl_\theta(A))$.

(ii) $\Rightarrow$ (i): Let F be θ-closed in X. Then $F = Cl_\theta(F)$. This means that $f(F) = f(Cl_\theta(F))$. By assumption, we have $f(F) \subseteq Cl_{\theta^\omega}(f(F)) \subseteq f(Cl_\theta(F)) = f(F)$. Thus, $Cl_{\theta^\omega}(f(F)) = f(F)$, which means $f(F)$ is θ^ω-closed in Y. Therefore, f is θ^ω-closed on X. □

Theorem 2.24. *Let X and Y be a topological spaces and $f : X \to Y$ be a bijective function. Then*

(i) *f is strongly θ^ω-open if and only if f is strongly θ^ω-closed.*

(ii) *f is θ^ω-open if and only if f is θ^ω-closed.*

Proof. (i): Suppose that f is strongly θ^ω-open on X and let G be closed in X. Then, $f(X \setminus G)$ is θ^ω-open in Y. Since f is bijective, $f(X \setminus G) = Y \setminus f(G)$ which is θ^ω-open in Y. Hence, $f(G)$ is θ^ω-closed in Y.

Conversely, suppose that f is strongly θ^ω-closed on X and let A be open in X. Thus, $f(X \setminus A)$ is θ^ω-closed in Y. Since f is bijective, $f(X \setminus A) = Y \setminus f(A)$ is θ^ω-closed in Y. Thus, $f(A)$ is θ^ω-open in Y.

(ii): Suppose that f is θ^ω-open on X and let G be θ-closed in X. Then, $f(X \setminus G)$ is θ^ω-open in Y. Since f is bijective, $f(X \setminus G) = Y \setminus f(G)$ which is θ^ω-open in Y. Hence, $f(G)$ is θ^ω-closed in Y.

Conversely, Suppose that f is θ^ω-closed on X and let A be θ-open in X. Thus, $f(X \setminus A)$ is θ^ω-closed in Y. Since f is bijective, $f(X \setminus A) = Y \setminus f(A)$ is θ^ω-closed in Y. Thus, $f(A)$ is θ^ω-open in Y. □

3 θ^ω-Continuous Functions and Other Versions of Continuity

This section characterizes the concepts of θ^ω-continuous functions and strongly θ^ω-continuous functions. Moreover, relationships of these concepts to the other well-known versions of continuity are described.

Definition 3.1. Let X and Y be topological spaces. A function $f : X \to Y$ is *strongly θ^ω-continuous on X* if $f^{-1}(A)$ is θ^ω-open in X for every open set A in Y.

Definition 3.2. Let X and Y be topological spaces. A function $f : X \to Y$ is *θ^ω-continuous on X* if $f^{-1}(A)$ is θ^ω-open in X for every θ-open set A in Y.

Theorem 3.3. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following are equivalent:*

(i) *f is θ^ω-continuous on X*

(ii) *$f^{-1}(F)$ is θ^ω-closed in X for each θ-closed subset F in Y*

(iii) *For every $x \in X$ and every open set V in Y containing $f(x)$, there exists an ω-open set U containing x such that $f(Cl_\omega(U)) \subseteq Cl(V)$.*

(iv) *$f(Cl_{\theta^\omega}(A)) \subseteq Cl_\theta(f(A))$ for each $A \subseteq X$.*

(v) *$Cl_{\theta^\omega}(f^{-1}(B)) \subseteq f^{-1}(Cl_\theta(B))$ for each $B \subseteq Y$.*

(vi) *$f^{-1}(V) \subseteq Int_{\theta^\omega}(f^{-1}(Cl(V)))$ for each open set $V \subseteq Y$.*

(vii) $Cl_{\theta^\omega}(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ *for each open set* $V \subseteq Y$.

Proof. (ii) $\Rightarrow$ (i): Suppose that (ii) holds. Let G be θ-open in Y. Then $Y \setminus G$ is θ-closed in Y. By assumption, $f^{-1}(Y \setminus G) = X \setminus f^{-1}(G)$ is θ^ω-closed in X. It follows that $f^{-1}(G)$ is θ^ω-open in X. Accordingly, f is θ^ω-continuous on X.

(i) $\Rightarrow$ (iii): Suppose that (i) holds. Let $x \in X$ and let V be an open set in Y containing $f(x)$. This means that there exists an open set O in Y containing $f(x)$ such that $O \subseteq V$. It follows that $Cl(O) \subseteq Cl(V)$. Thus, $f(x) \in Int_\theta(Cl(V))$. Since f is θ^ω-continuous and $Int_\theta(Cl(V))$ is θ-open in Y, $f^{-1}(Int_\theta(Cl(V)))$ is θ^ω-open in X. Hence, by Definition 2.1, there exists an ω-open set $U \ni x$ such that $Cl_\omega(U) \subseteq f^{-1}(Int_\theta(Cl(V)))$. Accordingly, $f(Cl_\omega(U)) \subseteq f(f^{-1}(Int_\theta(Cl(V)))) \subseteq Int_\theta(Cl(V)) \subseteq Cl(V)$.

(iii) $\Rightarrow$ (iv): Assume that (iii) holds. Let $A \subseteq X$ and $x \in Cl_{\theta^\omega}(A)$. Let O be an open set in Y with $f(x) \in O$. By assumption, there exists an ω-open set $V \ni x$ such that $f(Cl_\omega(V)) \subseteq Cl(O)$. Since $x \in Cl_{\theta^\omega}(A)$, then by Theorem 2.14 (xiv), $Cl_\omega(V) \cap A \neq \varnothing$. It follows that $\varnothing \neq f(Cl_\omega(V) \cap A) \subseteq f(Cl_\omega(V)) \cap f(A) \subseteq Cl(O) \cap f(A)$. Then, $f(x) \in Cl_\theta(f(A))$ so that $x \in f^{-1}(Cl_\theta(f(A)))$. Hence, $f(Cl_{\theta^\omega}(A)) \subseteq Cl_\theta(f(A))$.

(iv) $\Rightarrow$ (v): Let $B \subseteq Y$ and let $A = f^{-1}(B) \subseteq X$. Then $f(A) = f(f^{-1}(B)) \subseteq B$. By assumption, $f(Cl_{\theta^\omega}(A)) \subseteq Cl_\theta(f(A))$. Hence, $Cl_{\theta^\omega}(f^{-1}(B)) = Cl_{\theta^\omega}(A) \subseteq f^{-1}(f(Cl_{\theta^\omega}(A))) \subseteq f^{-1}(Cl_\theta(f(A))) \subseteq f^{-1}(Cl_\theta(B))$.

(v) $\Rightarrow$ (ii): Let F be a θ-closed subset in Y. Since F is θ-closed, it follows that $F = Cl_\theta(F)$. By assumption,

$$Cl_{\theta^\omega}(f^{-1}(F)) \subseteq f^{-1}(Cl_\theta(F)) = f^{-1}(F)$$

Since $f^{-1}(F) \subseteq Cl_{\theta^\omega}(f^{-1}(F))$, it follows that $Cl_{\theta^\omega}(f^{-1}(F)) = f^{-1}(F)$. Hence, $f^{-1}(F)$ is θ^ω-closed in X.

(iii) $\Rightarrow$ (vi): Suppose that (iii) holds. Let V be an open set in Y and let $x \in f^{-1}(V)$. Then $f(x) \in f(V)$. By assumption, there exists an ω-open set U containing x such that $f(Cl_\omega(U)) \subseteq Cl(V)$. This means that $x \in U \subseteq Cl_\omega(U) \subseteq f^{-1}(Cl(V))$. Thus, by Remark 2.14 (xiii), $x \in Int_{\theta^\omega}(f^{-1}(Cl(V)))$. Hence, (vi) holds.

(vi) $\Rightarrow$ (vii): Suppose that (vi) holds. Let V be an open set in Y and let $x \in Cl_{\theta^\omega}(f^{-1}(V))$. Suppose on the contrary that $x \notin f^{-1}(Cl(V))$. Then $f(x) \notin Cl(V)$. This means that there exists an open set G with $f(x) \in G$ such that $G \cap V = \varnothing$. Since both G and V are open, $Cl(G) \cap V = \varnothing$ so that $f^{-1}(Cl(G)) \cap f^{-1}(V) = \varnothing$. By assumption, $x \in f^{-1}(G) \subseteq Int_{\theta^\omega}(f^{-1}(Cl(G)))$. This implies that there exists an ω-open set U containing x such that $Cl_\omega(U) \subseteq f^{-1}(Cl(G))$. Hence, $Cl_\omega(U) \cap f^{-1}(V) = \varnothing$. By Remark 2.14 (xiv), $x \notin Cl_{\theta^\omega}(f^{-1}(V))$, a contradiction. Hence, $x \in f^{-1}(Cl(V))$ so that (vii) holds.

(vii) $\Rightarrow$ (iii): Assume that (vii) holds. Let $x \in X$ and let V be an open set in Y with $f(x) \in V$. Then $V \cap (Y \setminus Cl(V)) = \varnothing$ so that $f(x) \notin Cl(Y \setminus Cl(V))$. This means that $x \notin f^{-1}(Cl(Y \setminus Cl(V)))$. By assumption, $x \notin Cl_{\theta^\omega}(f^{-1}(Cl(Y \setminus Cl(V))))$. By Remark 2.14 (xiv), there exists an ω-open set $U \ni x$ such that

$$Cl_\omega(U) \cap f^{-1}(Y \setminus Cl(V)) = Cl_\omega(U) \cap (X \setminus f^{-1}(Cl(V))) = \varnothing$$

Thus, $f(Cl_\omega(U)) \subseteq Cl(V)$. Therefore, (iii) holds. □

Theorem 3.4. *Let* X *be a topological space and* V *be an open subset of* X. *Then*

(i) $Int(Cl(V))$ *is regular open; and*

(ii) $Cl(V)$ *is regular closed*

Proof. (i): Let $A = Int(Cl(V))$. Then $A = Int(Cl(V)) \subseteq Cl(V)$ so that $Cl(A) \subseteq Cl(V)$. Thus, $Int(Cl(A)) \subseteq Int(Cl(V)) = A$. Conversely, note that $A \subseteq Cl(A)$. Since A is open, $A = Int(A) \subseteq Int(Cl(A))$. Consequently, $Int(Cl(V))$ is regular open.

(ii): Let $B = Cl(V)$. Then, $V \subseteq Cl(V) = B$. Since V is open, $V = Int(V) \subseteq Int(B)$ so that $B = Cl(V) \subseteq Cl(Int(B))$. Conversely, note that $Int(B) \subseteq B$. Since B is closed, $Cl(Int(B)) \subseteq Cl(B) = B$. Accordingly, $Cl(V)$ is regular closed. □

Theorem 3.5. *Let X and Y be topological spaces and let $f : X \to Y$ be a function.*

(i) *If f is strongly θ^ω-continuous, then f is ω-continuous.*

(ii) *If f is ω-continuous, then f is almost ω-continuous.*

(iii) *If f is almost ω-continuous, then f is θ^ω-continuous.*

(iv) *If f is θ^ω-continuous, then f is weakly ω-continuous.*

Proof. (i): Suppose that f is strongly θ^ω-continuous. Then for every open set A in Y, $f^{-1}(A)$ is θ^ω-open in X. By Theorem 2.2 (ii), every θ^ω-open is ω-open. Hence, $f^{-1}(A)$ is ω-open. Thus, f is ω-continuous.

(ii): Let f be ω-continuous on X. Let V be a regular open set in Y. Then, V is open in Y. By assumption, $f^{-1}(V)$ is ω-open in X. Hence, by [16, Theorem 2.2], f is almost ω-continuous on X.

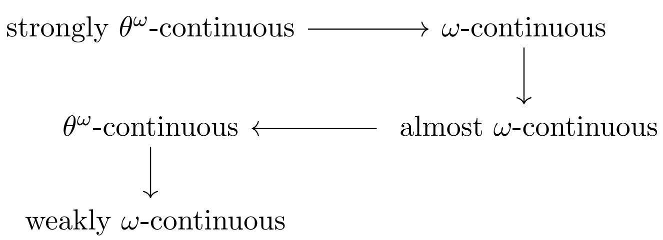
(iii): Suppose that f is almost ω-continuous. Let $x \in X$ and let V be open in Y containing $f(x)$. Then $f^{-1}(Int(Cl(V))$ is ω-open in X by [16, Theorem 2.2]. Since $Cl(V)$ is regular closed by Theorem 3.4, $f^{-1}(Cl(V))$ is ω-closed in X by [16, Theorem 2.2]. Let $U = f^{-1}(Int(Cl)V)$ which contains x. Then

$$Cl_\omega(U) = Cl_\omega f^{-1}(Int(Cl(V)) \subseteq Cl_\omega(f^{-1}(Cl(V)) = f^{-1}(Cl(V)).$$

Consequently, $f(Cl_\omega(U)) \subseteq Cl(V)$. Then by Theorem 3.3 (iii), f is θ^ω-continuous.

(iv): Let f be θ^ω-continuous. Let $x \in X$ and let V be open in Y containing $f(x)$. Then by Theorem 3.3, there exists an ω-open set $U \ni x$ such that $f(Cl_\omega(U)) \subseteq Cl(V)$. Since $U \subseteq Cl_\omega(U)$, it follows that $f(U) \subseteq f(Cl_\omega(U)) \subseteq Cl(V)$. Thus, f is weakly ω-continuous. □

Remark 3.6. The following diagram holds for a function $f : X \to Y$.



Except for θ^ω-continuity and weakly ω-continuity, the following examples show that the implications above are not reversible.

Example 3.7. Let $X = [\sqrt{2}, 2)$ and $Y = \{a, b\}$ be two topological spaces with respective topologies $\mathcal{T}_X = \{\varnothing, [\sqrt{2}, 2), \mathbb{Q}^c \cap [\sqrt{2}, 2)\}$ and $\mathcal{T}_Y = \{\varnothing, Y, \{a\}\}$. Define $f : X \to Y$ as follows

$$f(x) = \begin{cases} a, & \text{if } x \in \mathbb{Q}^c \cap [\sqrt{2}, 2) \\ b, & \text{if } x \in \mathbb{Q} \cap [\sqrt{2}, 2). \end{cases}$$

Note that $f^{-1}(\{a\}) = \mathbb{Q}^c \cap [\sqrt{2}, 2)$ and $f^{-1}(Y) = X$ are ω-open in X. However, f is not strongly θ^ω-continuous since $f^{-1}(\{a\}) = \mathbb{Q}^c \cap [\sqrt{2}, 2)$ is not θ^ω-open, by Example 2.7.

Example 3.8. Let $X = \mathbb{R}$ and $Y = \{a, b, c\}$ be two topological spaces with respective topologies $\mathcal{T}_X = \{\varnothing, \mathbb{R}, \mathbb{Z}\}$ and $\mathcal{T}_Y = \{\varnothing, Y, \{a\}\}$.

First, we show that $\{a\}$ is not regular open in Y. Note that $Int(Cl(\{a\})) = Int(Y) = Y \neq \{a\}$. Hence, $\{a\}$ is not regular open. It follows that the only regular open sets in Y are $\varnothing$ and Y.

Next, we show that $\mathbb{Q}$ is not ω-open in X. Suppose that $\mathbb{Q}$ is ω-open. Then for all $x \in \mathbb{Q}$, there exists an open set $U \ni x$ such that $U \setminus \mathbb{Q}$ is countable. If $x \in \mathbb{Q} \setminus \mathbb{Z}$, the only open set containing x is $\mathbb{R}$. But $\mathbb{R} \setminus \mathbb{Q}$ is uncountable, a contradiction. Thus, $\mathbb{Q}$ is not ω-open.

Define $f : X \to Y$ as follows

$$f(x) = \begin{cases} a, & \text{if } x \in \mathbb{Q} \\ b, & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then, f is almost ω-continuous since $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$ are ω-open. However, f is not ω-continuous since $f^{-1}(\{a\}) = \mathbb{Q}$ is not ω-open in X.

Example 3.9. Let $X = \mathbb{R}$ and $Y = \{a, b, c, d\}$ be two topological spaces with respective topologies $\mathcal{T}_X = \{\varnothing, \mathbb{R}, \mathbb{Q}^c\}$ and $\mathcal{T}_Y = \{\varnothing, Y, \{a\}, \{c, d\}, \{a, c, d\}\}$.

We will show first that $\mathbb{N}$ is not ω-open in X. Suppose that $\mathbb{N}$ is ω-open in X. Then, for all $x \in \mathbb{N}$, there exists an open set $U \ni x$ such that $U \setminus \mathbb{N}$ is countable. Note further that the only open set U containing $x \in \mathbb{N}$ is $\mathbb{R}$. But $\mathbb{R} \setminus \mathbb{N}$ is uncountable, a contradiction. Thus, $\mathbb{N}$ is not ω-open.

Define $f : X \to Y$ as follows

$$f(x) = \begin{cases} a, & \text{if } x \in \mathbb{N} \\ b, & \text{if } x \notin \mathbb{N} \end{cases}$$

Observe that the only θ open sets in Y are $\varnothing$ and Y. Also, $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = \mathbb{R}$ are θ^ω-open. Thus f is θ^ω-continuous on X. Next, note that $\{a\}$ is a regular open set since $\{a\} = Int(Cl(\{a\})) = Int(\{a, b\}) = \{a\}$. However, $f^{-1}(\{a\}) = \mathbb{N}$ is not ω-open in X. Therefore, by [16, Theorem 2.2], f is not almost ω-continuous.

Theorem 3.10. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following are equivalent:*

(i) *f is strongly θ^ω-continuous on X.*

(ii) *$f^{-1}(F)$ is θ^ω-closed in X for each closed subset F in Y.*

(iii) *$f^{-1}(B)$ is θ^ω-open in X for each (subbasic) basic open set B in Y.*

(iv) *For every $x \in X$ and every open set V in Y containing $f(x)$, there exists an ω-open set U containing x such that $f(Cl_\omega(U)) \subseteq V$.*

(v) *$f(Cl_{\theta^\omega}(A)) \subseteq Cl(f(A))$ for each $A \subseteq X$.*

(vi) *$Cl_{\theta^\omega}(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for every $B \subseteq Y$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that f is strongly θ^ω-continuous on X. Let F be a closed set in Y. Then $Y \setminus F$ is open in Y. Since f is θ^ω-continuous, then $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is θ^ω-open in X. Thus, $f^{-1}(F)$ is θ^ω-closed in X.

(ii) $\Rightarrow$ (i): Assume that $f^{-1}(F)$ is θ^ω-closed in X for each closed set F in Y. Let G be an open set in Y. Then, $Y \setminus G$ is a closed set in Y so that $f^{-1}(Y \setminus G) = X \setminus f^{-1}(G)$ is θ^ω-closed in X. This means that $f^{-1}(G)$ is θ^ω-open in X. Therefore, f is strongly θ^ω-continuous on X.

(i) $\Rightarrow$ (iii): Suppose that f is strongly θ^ω-continuous on X. Note that every (subbasic) basic open set is open. Thus, by assumption, (iii) holds.

(iii) $\Rightarrow$ (i) Let $f^{-1}(B)$ be θ^ω-open in X for each $B \in \beta$, where β is a basis for the topology in Y. Let G be an open set in Y. Then,

$$G = \bigcup\{B : B \in \beta^*\}$$

where $\beta^* \subseteq \beta$. It follows that $f^{-1}(G) = \bigcup\{f^{-1}(B) : B \in \beta^*\}$ Since the arbitrary union of θ^ω-open sets is θ^ω-open by Theorem 2.12, $f^{-1}(G)$ is θ^ω-open in X. It follows that f is strongly θ^ω-continuous on X.

(i) $\Rightarrow$ (iv): Suppose that f is strongly θ^ω-continuous on X. Let $x \in X$ and let V be an open set in Y such that V contains $f(x)$. Since f is strongly θ^ω-continuous, $f^{-1}(V)$ is θ^ω-open in X containing x. Thus, there exists an ω-open set U that contains x such that $Cl_\omega(U) \subseteq f^{-1}(V)$. Consequently, $f(Cl_\omega(U)) \subseteq f(f^{-1}(V)) \subseteq V$. Thus (iv) is satisfied.

(iv) $\Rightarrow$ (v): Suppose that (iv) holds. Let $A \subseteq X$ and $x \in Cl_{\theta^\omega}(A)$. Let O be an open set in Y containing $f(x)$. By assumption, there exists an ω-open set U containing x such that $f(Cl_\omega(U)) \subseteq O$. Since $x \in Cl_{\theta^\omega}(A)$, by Remark 2.14, $Cl_\omega(U) \cap A \neq \varnothing$. It follows that $\varnothing \neq f(Cl_\omega(U) \cap A) \subseteq f(Cl_\omega(U)) \cap f(A) \subseteq O \cap f(A)$. This implies that $f(x) \in Cl(f(A))$. Thus, $f(Cl_{\theta^\omega}(A)) \subseteq Cl(f(A))$.

(v) $\Rightarrow$ (vi): Assume that $f(Cl_{\theta^\omega}(A)) \subseteq Cl(f(A))$. Let $B \subseteq Y$ and $A = f^{-1}(B)$. Since $A = f^{-1}(B)$, then $Cl_{\theta^\omega}(f^{-1}(B)) = Cl_{\theta^\omega}(A)$. Hence,

$$Cl_{\theta^\omega}(f^{-1}(B)) \subseteq f^{-1}(f(Cl_{\theta^\omega}(A))) \subseteq f^{-1}(Cl(f(A))) \subseteq f^{-1}(Cl(B)).$$

(vi) $\Rightarrow$ (ii): Let F be a closed set in Y. Then, $F = Cl(F)$. By assumption,

$$Cl_{\theta^\omega}(f^{-1}(F)) \subseteq f^{-1}(Cl(F)) = f^{-1}(F) \subseteq Cl_{\theta^\omega}(f^{-1}(F)).$$

Since $f^{-1}(F) \subseteq Cl_{\theta^\omega}(f^{-1}(F))$, it follows that $Cl_{\theta^\omega}(f^{-1}(F)) = f^{-1}(F)$ so that $f^{-1}(F)$ is θ^ω-closed in X by Remark 2.14 (vii). □

Theorem 3.11. *Let X and Y be topological spaces and $f_A : X \to D$ the characteristic function of a subset A of X, where D is the set $\{0,1\}$ with the discrete topology. Then the following statements are equivalent:*

(i) *f_A is strongly θ^ω-continuous on X.*

(ii) *A is both θ^ω-open and θ^ω-closed.*

(iii) *f_A is θ^ω-continuous on X.*

Proof. (i) $\Rightarrow$ (ii): Let f_A be strongly θ^ω-continuous on X. Let $G_1 = \{1\}$ and $G_2 = \{0\}$. Then G_1 and G_2 are open in $\{0,1\}$. Hence, $f_A^{-1}(G_1) = A$ and $f_A^{-1}(G_2) = X \setminus A$ are θ^ω-open in X. Hence, A is both θ^ω-open and θ^ω-closed.

(ii) $\Rightarrow$ (i): Suppose that A is both θ^ω-open and θ^ω-closed. Let G be an open set in $\{0,1\}$. Then,

$$f_A^{-1}(G) = \begin{cases} \varnothing & \text{if } G = \varnothing \\ X & \text{if } G = \{0,1\} \\ A & \text{if } G = \{1\} \\ X \setminus A & \text{if } G = \{0\} \end{cases}$$

Hence, $f_A^{-1}(G)$ is θ^ω-open and so f_A is strongly θ^ω-continuous on X.

(i) $\Rightarrow$ (iii): Follows from Remark 3.6.

(iii) $\Rightarrow$ (ii): Let f_A be θ^ω-continuous on X. Since every set in D is θ-open, using similar argument in (i) $\Rightarrow$ (ii), (ii) holds. □

The following results are related to strongly θ^ω-continuous functions in the product space.

In the succeeding results, if $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ is a product space and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, we denote $A_{\alpha_1} \times \cdots \times A_{\alpha_n} \times \prod\{Y_\alpha : \alpha \notin K\}$ by $\langle A_{\alpha_1}, \cdots, A_{\alpha_n}\rangle$, where $K = \{\alpha_1, \cdots, \alpha_n\}$.

Theorem 3.12. *Let X be a topological space and $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space. A function $f : X \to Y$ is strongly θ^ω-continuous on X if and only if each coordinate function $p_\alpha \circ f$ is strongly θ^ω-continuous on X for every $\alpha \in \mathcal{A}$.*

Proof. Assume that f is strongly θ^ω-continuous on X. Let $\alpha \in \mathcal{A}$ and U_α be open in Y_α. Since p_α is continuous, then $p_\alpha^{-1}(U_\alpha)$ is open in Y. Thus,

$$f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$$

is θ^ω-open in X. Hence, $p_\alpha \circ f$ is strongly θ^ω-continuous for every $\alpha \in \mathcal{A}$.
Conversely, assume that each coordinate function $p_\alpha \circ f$ is strongly θ^ω-continuous. Let O_α be open in Y_α. Then $\langle O_\alpha\rangle$ is a subbasic open set in Y and $(p_\alpha \circ f)^{-1}(O_\alpha) = f^{-1}(p_\alpha^{-1}(O_\alpha)) = f^{-1}(\langle O_\alpha\rangle)$ is θ^ω-open in X. Thus, by Definition 3.1, f is strongly θ^ω-continuous on X. □

Corollary 3.13. *Let X be a topological space, $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space, and $f_\alpha : X \to Y_\alpha$ be a function for each $\alpha \in \mathcal{A}$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x)\rangle$. Then f is strongly θ^ω-continuous on X if and only if each f_α is strongly θ^ω-continuous on X for each $\alpha \in \mathcal{A}$.*

Proof. Let $\alpha \in \mathcal{A}$ and $x \in X$. Then we have

$$(p_\alpha \circ f)(x) = p_\alpha(f(x)) = p_\alpha(\langle f_\alpha(x)\rangle) = f_\alpha(x)$$

Thus, $p_\alpha \circ f = f_\alpha$ for each $\alpha \in \mathcal{A}$. The result follows from Theorem 3.12. □

4 θ^ω-Connected Space

This section provides a characterization of θ^ω-connectedness and investigate its relationship to connected, θ-connected, and ω-connected topological spaces.

Definition 4.1. A topological space X is said to be a θ^ω-*connected* if it is not the union of two nonempty disjoint θ^ω-open sets. Otherwise, X is said θ^ω-*disconnected*. A subset B of X is θ^ω-connected if it is θ^ω-connected as a subspace of X.

Example 4.2. Let $X = \mathbb{R}$ with topology $\mathfrak{T} = \{\varnothing, \mathbb{R}, \mathbb{Q}^c\}$. Note that if U is ω-open, then U must be uncountable since $\mathbb{R}$ and $\mathbb{Q}^c$ are uncountable. Let A be a θ^ω-open set in $\mathbb{R}$. We will show that $A^c := \mathbb{R}\setminus A$ is not θ^ω-open so that $(\mathbb{R}, \mathfrak{T})$ is θ^ω-connected. Suppose that A^c is θ^ω-open. Then for all $x \in A$, there exists an ω-open set U containing x such that $U \subseteq Cl_\omega(U) \subseteq A$. Since U is ω-open, either $\mathbb{R} \setminus U$ is countable or $\mathbb{Q}^c \setminus U$ is countable. If $\mathbb{Q}^c \setminus U$ is countable, then $\mathbb{R} \setminus U$ is countable since $\mathbb{R} \setminus U \subseteq \mathbb{Q} \cup (\mathbb{Q}^c \setminus U)$. Either case, $\mathbb{R} \setminus U$ is countable. Observe that $U \subseteq Cl_\omega(U) \subseteq A$ implies that $\mathbb{R} \setminus A \subseteq \mathbb{R} \setminus U$ so that $\mathbb{R} \setminus A$ is also countable. It follows that $\mathbb{R} \setminus A$ is not ω-open. Thus, $\mathbb{R} \setminus A$ is not θ^ω-open, by Theorem 2.2 (ii). Therefore, $(\mathbb{R}, \mathfrak{T})$ is θ^ω-connected.

Theorem 4.3. *Let X be a topological space. Then the following statements are equivalent.*

(i) *X is θ^ω-connected.*

(ii) *The only subsets of X that are both θ^ω-open and θ^ω-closed are $\varnothing$ and X.*

(iii) *No θ^ω-continuous function $f : X \to \mathcal{D}$ is surjective.*

(iv) *No strongly θ^ω-continuous function $f : X \to \mathcal{D}$ is surjective.*

Proof. (i) $\Rightarrow$ (ii): Assume that X is θ^ω-connected and $A \subseteq X$. Let A be both θ^ω-open and θ^ω-closed. Then $X \setminus A$ is both θ^ω-open and θ^ω-closed. Note that $X = A \cup (X \setminus A)$. Since X is θ^ω-connected, by assumption, so A is either $\varnothing$ or X.

(ii) $\Rightarrow$ (iii): Suppose that $\varnothing$ and X are the only subsets of X that are both θ^ω-open and θ^ω-closed and let $f : X \to \mathcal{D}$ be a θ^ω-continuous surjection. Then $f^{-1}(\{0\}) \neq \varnothing, X$. Note that every set in $\mathcal{D}$ is θ-open. Also, $\{0\}$ is both θ-open and θ-closed in $\mathcal{D}$. Then $f^{-1}(\{0\})$ is both θ^ω-open and θ^ω-closed in X, a contradiction.

(iii) $\Rightarrow$ (iv): Suppose that no θ^ω-continuous function $f : X \to \mathcal{D}$ is surjective. Since every strongly θ^ω-continuous is θ^ω-continuous. It follows that no strongly θ^ω-continuous function $f : X \to \mathcal{D}$ is surjective.

(iv) $\Rightarrow$ (i): Suppose no strongly θ^ω-continuous function $f : X \to \mathcal{D}$ is surjective. Then let $X = A \cup B$, where A and B are disjoint nonempty θ^ω-open sets. Then A and B are also θ^ω-closed sets. Consider the characteristic function $f_A : X \to \mathcal{D}$ of $A \subseteq X$. By Theorem 3.11, f_A is strongly θ^ω-continuous, a contradiction. Thus, X is θ^ω-connected. □

Theorem 4.4. *Let X be a topological space. Then X is ω-connected if and only if X is θ^ω-connected.*

Proof. Suppose that X is ω-connected. Thus, X cannot be expressed as the union of two nonempty disjoint ω-open sets. Since every θ^ω-open is ω-open, by Theorem 2.2 (ii), X cannot be the union of two nonempty disjoint θ^ω-open sets. Thus, X is θ^ω-connected.

Conversely, assume that X is θ^ω-connected. Suppose on the contrary that X is ω-disconnected. Then $X = A \cup B$, where A and B are two disjoint ω-open sets. This means that $A = X \setminus B$ and $B = X \setminus A$ are ω-closed sets. By Lemma 2.4, A and B are θ^ω-open. Hence, X is θ^ω-disconnected, a contradiction. Thus, X is ω-connected. □

In [12], θ-connected space and connected space are equivalent. Moreover, in [14, Remark 4 (iii)], every ω-connected space is connected space. Then we have the following corollary.

Corollary 4.5. *Let X be a topological space. If X is θ^ω-connected, then X is connected.*

In view of Theorem 4.4 and Corollary 4.5, we have the following remark.

Remark 4.6. The following diagram holds for a subset of topological space.

$$\begin{array}{ccc}
\omega\text{-connected} & \longleftrightarrow & \theta^\omega\text{-connected} \\
\downarrow & & \downarrow \\
\text{connected} & \longleftrightarrow & \theta\text{-connected}
\end{array}$$

The converse of Remark 4.6, is not necessarily true.

Consider $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a\}\}$. It is not difficult to see that $(X, \mathcal{T})$ is connected. However, $(X, \mathcal{T})$ is θ^ω-disconnected since $\{a\}$ and $\{b, c\}$ are θ^ω-open sets and $\{a\} \cup \{b, c\} = X$.

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