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### Roman Domination Number of the Join and Corona of Graphs

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**Abstract:** A *Roman dominating function* on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of $f$ is $w(f) = \sum_{v \in V} f(v)$. The *Roman domination number* is the minimum weight of a Roman dominating function in $G$.

In this paper, a sharp upper bound of the Roman domination number of the join of two arbitrary graphs is given. In addition, the Roman domination number of the corona of two arbitrary graphs is also given.
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# 1 Introduction

A lot of mathematicians are into the study of domination in graphs, so that there are already so many varieties of it. Some even have historical motivations. For instance, the Roman dominating function. As mentioned in the article Roman Domination in Graphs [1], Constantine the Great, the Emperor of Rome, issued an order in the 4th century A.D. for the protection of his empire. He decreed that any city without a legion stationed to secure itself must be a neighbor of another city having two legions. If the former were attacked, then the latter could deploy a legion to protect it without itself becoming vulnerable. Cockayne et al. [1] studied the graph theoretic properties of the Roman domination.

The *complement* of a graph $G$, denoted by $\overline{G}$, is a graph with the same vertex set as $G$ and where two distinct vertices are adjacent if and only if they are not adjacent in $\overline{G}$.

The *cycle* $C_n = [v_1, v_2, \ldots, v_n]$, $n \geq 3$, is the graph with vertices $v_1$, $v_2$, $\ldots$, $v_n$ and edges $v_1v_2$, $v_2v_3$, $\ldots$, $v_{n-1}v_n$, $v_nv_1$.

Let $X$ and $Y$ be sets. The *disjoint union* of $X$ and $Y$, denoted by $X \dot{\cup} Y$, is found by combining the elements of $X$ and $Y$, treating all elements to be distinct. Thus, $|X \dot{\cup} Y| = |X| + |Y|$

The *join* of two graphs $G$ and $H$, denoted by $G+H$, is the graph with vertex-set $V(G+H) = V(G) \dot{\cup} V(H)$ and edge-set $E(G + H) = E(G) \dot{\cup} E(H) \dot{\cup} \{uv : u \in V(G), v \in V(H)\}$.

The *fan* $F_n$ of order $n+1$ is the graph obtained from the path $P_n$ by adding a new vertex, say $x_0$, and joining $x_0$ by an edge to each of the $n$ vertices of $P_n$. Thus, for $n \geq 1$, $F_n = K_1 + P_n$. The *generalized fan* $F_{m,n}$ of order $m+n$ is the join $\overline{K}_m + P_n$, of graphs $P_n$ and $\overline{K}_m$.

The *wheel* $W_n$ of order $n+1$ is the graph obtained from the cycle $C_n$ by adding a new vertex, say $x_0$, and joining $x_0$ by an edge to each of the $n$ vertices of $C_n$. Thus, for $n \geq 3$, $W_n = K_1 + C_n$. The *generalized wheel* $W_{m,n}$ of order $m+n$ is the join $\overline{K}_m + C_n$, of $\overline{K}_m$ and $C_n$.

Let $G$ be a graph of order $n$. The *corona* $G \circ H$ of two graphs $G$ and $H$ is the graph obtained by taking one copy of $G$ and $n$ copies of $H$, and then joining the *ith* vertex of $G$ to every vertex of the *ith* copy of $H$. The symbol $w + H^w$ denotes the subgraph of $G \circ H$ formed by joining the *wth* vertex of $G$ to every vertex of the *wth* copy of $H$.

A set $S \subseteq V(G)$ is a *dominating set* of $G$ if for every $x \in V(G)\backslash S$, there exists $y \in S$ such that $xy \in E(G)$, that is, $N_G[S] = S \cup N_G(S) = V(G)$, where

$$N_G(S) = \{v \in V(G) : uv \in E(G) \; for \; some \; u \in S\}.$$

A *Roman dominating function* on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of $f$ is $w(f) = \sum_{v \in V} f(v)$. The *Roman domination number* is the minimum weight of a Roman dominating function in $G$.

**Remark 1.1** The number or weight assigned to a vertex may be viewed as the number of guards or legion in that vertex.

# 2 Results

This section presents the results in this study. Lemma 2.1 will be used in the proof of some results.

**Lemma 2.1** *Let $G$ be a graph with $|V(G)| > 1$. Then $\gamma_R(G) > 1$.*

*Proof*: Suppose $\gamma_R(G) = 1$ and let $f$ be a Roman dominating function in $G$ such that $w(f) = 1$. Then $f(y) = 1$ for some $y \in V(G)$ and $f(x) = 0$ for all $x \in V(G)\backslash\{y\}$. Since $|V(G)| > 1$, $|V(G)\backslash\{y\}| \geq 1$. Let $z \in V(G)\backslash\{y\}$. Since $f(z) = 0$, there exists $x \in V(G)\backslash\{z, y\}$ such that $f(x) = 2$. This gives a contradiction. □

The following result characterizes graphs which have Roman domination number equal to 2.

**Theorem 2.2** *Let $G$ be any graph of order $n > 1$. Then $\gamma_R(G) = 2$ if and only if either $\overline{K_2}$ or $G$ has a subgraph $K_{1,n-1}$.*

*Proof*: If $G = \overline{K_2}$ and $f$ is a Roman dominating function in $G$, then $f(x) = 1$ for all $x \in G$. Thus, $w(f) = 2$. Since $f$ is arbitrary, $\gamma_R(G) = 2$. Suppose that $G$ has a subgraph $K_{1,n-1}$ with partite sets $\{v\}$ and $\{u_1, u_1, \ldots, u_1\}$. Define $f : V(G) \to \{0, 1, 2\}$ by $f(v) = 2$ and $f(u_i) = 0$, for all $i = 1, 2, \ldots, n-1$. Then $f$ is a Roman dominating function in $G$. Thus, $\gamma_R(G) \leq 2$. Since, $|V(G)| > 1$, by Lemma 2.1, $\gamma_R(G) > 1$. Therefore, $\gamma_R(G) = 2$.

Conversely, suppose that $\gamma_R(G) = 2$. Clearly, $|V(G)| \geq 2$. Suppose that $|V(G)| = 2$. Then either $G = K_2$ or $G = \overline{K_2}$. Viewing $K_2$ as $K_{1,1}$, we are done. So we assume that $|V(G)| > 2$, and let $f$ be a Roman dominating function in $G$ of minimum weight. If $f(v) \geq 1$ for all $v \in V(G)$, then $\gamma_R(G) > 2$, a contradiction. Thus, there exists $v \in V(G)$ such that $f(v) = 0$. Consequently, there exists $u_0 \in V(G)$ such that $u_0 v \in E(G)$ and $f(u_0) = 2$. Since $w(f) = 2 = \sum_{v \in V(G)} f(v)$, $f(v) = 0$ for all $v \in V(G)\backslash\{u_0\}$. This means that $u_0 v \in E(G)$ for all $v \in V(G)\backslash\{u_0\}$. Therefore, $K_{1,n-1}$ is a subgraph of $G$. □

The next result gives the Roman domination number of the join of two arbitrary graphs.

**Corollary 2.3** *Let $G$ be a graph. Then $\gamma_R(K_1 + G) = 2$.*

*Proof*: Let $G$ be a graph of order $n$. Then $\overline{K_n}$ is a subgraph of $G$ so that $K_{1,n}$ is a subgraph of $K_1 + G$. Thus, by Theorem 2.2, $\gamma_R(K_1 + G) = 2$. □

**Corollary 2.4** *Let $G$ be a graph. Then $\gamma_R(K_n + G) = 2$ for all positive integers $n$.*

*Proof*: Let $n \in \mathbb{N}$. Then, by Corollary 2.3, $\gamma_R(K_n + G) = \gamma_R(K_1 + (K_{n-1} + G)) = 2$. $\square$

**Theorem 2.5** *Let $G$ and $H$ be a graphs. Then $\gamma_R(G + H) \leq 4$.*

*Proof*: Let $u \in V(G)$ and $u \in V(H)$ and define $f : V(G + H) \to \{0, 1, 2\}$ by $f(u) = 2 = f(v)$ and $f(x) = 0$ for all $x \in V(G + H)\backslash\{u, v\}$. Then $f$ is a Roman dominating function in $G + H$. Therefore, $\gamma_R(G + H) \leq 4$. $\square$

**Theorem 2.6** *Let $G$ and $H$ be a graphs of order $m$ and $n$, respectively.*

1. *$\gamma_R(G + H) = 2$ if and only if either there exists $u \in V(G)$ such that $ux \in E(G)$ for all $x \in V(G)\backslash\{u\}$, or there exists $u \in V(H)$ such that $ux \in E(H)$ for all $x \in V(H)\backslash\{u\}$.*
2. *If either $G = \overline{K_2}$ and $m \geq 2$, or $H = \overline{K_2}$ and $n \geq 2$, then, $\gamma_R(G + H) = 3$.*

*Proof*: (1) Follows from Theorem 2.2. (2) Suppose that $G = \overline{K_2}$ and $m \geq 2$ Let $u, v \in V(G)$. Define $f : V(G+H) \to \{0, 1, 2\}$ by $f(u) = 2, = f(v) = 1$ and $f(x) = 0$ for all $x \in V(G+H)\backslash\{u, v\}$. Then $f$ is a Roman dominating function in $G + H$. Therefore, $\gamma_R(G + H) \leq 3$. But from (1) and Lemma 2.1, $\gamma_R(G + H) > 2$. Hence, $\gamma_R(G + H) = 3$. Similarly, suppose that $H = \overline{K_2}$ and $n \geq 2$. Let $u, v \in V(H)$. Define $f : V(G+H) \to \{0, 1, 2\}$ by $f(u) = 2, = f(v) = 1$ and $f(x) = 0$ for all $x \in V(G + H)\backslash\{u, v\}$. Then $f$ is a Roman dominating function in $G + H$. Therefore, $\gamma_R(G + H) \leq 3$. But from (1) and Lemma 2.1, $\gamma_R(G + H) > 2$. Hence, $\gamma_R(G + H) = 3$. $\square$

We recall the *Pigeonhole Principle*. We shall use it in the next theorem. Chong et al.[1] stated the following for the Pigeonhole Principle. Let $k$ and $n$ be any two positive integers. If at least $kn + 1$ objects are distributed among $n$ boxes, then one of the boxes must contain at least $k + 1$ objects. In particular, if at least $n + 1$ objects are to be put into $n$ boxes, then one of the boxes must contain at least two objects.

**Theorem 2.7** *Let $G$ and $H$ be graphs. Then $\gamma_R(G \circ H) = 2\,|V(G)|$.*

*Proof*: Let $G$ and $H$ be a graphs. Let $f : V(G \circ H) \to \{0, 1, 2\}$ be defined by $f(x) = 2$ for all $x \in V(G)$, and $f(z) = 0$ for all $z \in V(G \circ H)\backslash V(G)$. Then by the adjacency in $G \circ H$, $f$ is a Roman dominating function. Therefore, $\gamma_R(G \circ H) \leq 2\,|V(G)|$. Suppose $\gamma_R(G \circ H) < 2\,|V(G)|$. Let $f$ is a Roman dominating function in $G \circ H$ with $w(f) < 2\,|V(G)|$. Then by *Pigeon Hole Principle*, there exist a $w \in V(G)$ such that in $w + H^w$, $f(x) = 1$ for some $x \in w + H^w$ and $f(y) = 0$ for all $y \in w + H^w\backslash\{x\}$. Let $z \in w + H^w$ with $z \neq x$. Then $f(z) = 0$ and $z$ is not adjacent to any vertex $v \in V(G \circ H)$ with $f(v) = 2$. This implies $f$ is not a Roman dominating function. This is a contradiction. Therefore, $\gamma_R(G \circ H) \geq 2\,|V(G)|$. Accordingly, $\gamma_R(G \circ H) = 2\,|V(G)|$. $\square$

**Corollary 2.8** *Let $F_n$ and $W_n$ be a fan of order $n + 1$ and a wheel of order $n + 1$, respectively. Then*

1. $\gamma_R(F_n) = 2$,
2. $\gamma_R(W_n) = 2$.

# References

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