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Weighted Improved Hardy-Sobolev Inequality on a Ball Domain

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Abstract: Let Ω be a bounded domain in $\mathbb{R}^n$ with $0 \in \Omega$ and $n \geq 2$. We consider the Hardy-Sobolev inequality

$$\int_\Omega |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u(x)^2}{|x|^2} dx \tag{0.1}$$

for any $u \in W_0^{1,2}(\Omega)$. When no weight function is involved, the improvement of (0.1) is already proven. In this paper[1], we shall investigate the weighted type of the improvement of (0.1) in a ball domain.

1 Introduction

The study of the minimal and the extremal solutions of the quasilinear elliptic equations gains much attention in the recent years because of its applications in Magnetic and Potential Theory. Under some conditions, the analysis will start on the establishment of the existence of the minimal and the extremal solutions and then study their behaviors in the linearized quasilinear elliptic equations. To analyze the linearized equation at extremal solution, the classical Hardy-Sobolev inequality in not enough since it has only singularity at the origin. Hence we have to essentially improve the classical result by having a weight on both sides of the inequality. The study will focus on improving the classical result of A.L. Detalla [4] by having a weight function $|x|^\alpha$ on both sides of the equation where the domain is a ball centered at $x \in \mathbb{R}^n$ of radius ρ and this is denoted by $B_\rho(x)$ where $B_\rho(x) \subset \mathbb{R}^n$.

2 Known Results

Let Ω be a bounded domain in $\mathbb{R}^n$ with $0 \in \Omega$ and $n \geq 2$. For $1 < p < n$ the well known Hardy-Sobolev inequality

$$\int_\Omega |\nabla u(x)|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{u(x)^p}{|x|^p} dx. \tag{2.1}$$

holds for $u \in W_0^{1,p}(\Omega)$, where $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm

$$||u||_{1,p,\Omega} := \left(\int_\Omega |u(x)|^p dx + \int_\Omega |\nabla u(x)|^p dx\right)^{\frac{1}{p}}.$$

An improvement of inequality (2.1) involving one remainder term in the right hand side was proven by Adimurthi, Chaudhuri, and Ramaswamy [1] and this is given by the following results:

[1]2000 *Mathematics Subject Classification.* Primary 35J70; Secondary 35J60.
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1. Noncritical case $(1 < p < n)$**:** Let $R \geq \sup_{\Omega}\left(|x|e^{\frac{2}{p}}\right)$. Then there exist $K > 0$ depending on n, p, and R such that for any $u \in W_0^{1,p}(\Omega)$

$$\int_{\Omega} |\nabla u(x)|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_{\Omega} \frac{|u(x)|^p}{|x|^p} dx + K \int_{\Omega} \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx \qquad (2.2)$$

where $\gamma \geq 2$.
2. Critical case $(p = n)$**:** Let $R \geq \sup_{\Omega}\left(|x|e^{\frac{2}{n}}\right)$. Then for any $u \in W_0^{1,n}(\Omega)$

$$\int_{\Omega} |\nabla u(x)|^n dx \geq \left(\frac{n-1}{n}\right)^n \int_{\Omega} \frac{|u(x)|^n}{|x|^n} \left(\log \frac{R}{|x|}\right)^{-n} dx \qquad (2.3)$$

For $p = 2$ an optimal improvement of the Hardy-Sobolev inequality with infinitely many terms was proven by Detalla, Horiuchi, and Ando [4] and is given by

$$\int_{\Omega} |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{u(x)^2}{|x|^2} dx + \frac{1}{4} \int_{\Omega} \frac{u(x)^2}{|x|^2} \Big[A_1(|x|)^{-2} + \left(A_1(|x|)A_2(|x|)\right)^{-2} + \cdots + \left(A_1(|x|)A_2(|x|)\cdots A_k(|x|)\right)^{-2}\Big] dx. \qquad (2.4)$$

for any $u \in W_0^{1,2}(\Omega)$ where $A_1(|x|) = \log \frac{R}{|x|}$ and $A_k(|x|) = \log A_{k-1}(|x|)$. Here $R \geq e_k \sup_{\Omega} |x|$ and $e_1 = e, \quad e_k = e^{e_{k-1}}$.

This study aims improve this classical result by having a weight on both sides of inequality (2.4) on a ball domain.

3 Result and Discussion

In this section we will introduce our result about the weighted type of inequality (2.4). The main results are as follows:

Theorem 3.1 *Let n, α and k be a positive integers and a ball $B_\rho \subset \Omega$ such that $n > 2$, $k \geq 1$ and $R \geq e_k \sup_{\Omega} |x|$. Then the inequality*

$$\int_{B_\rho} |\nabla u(x)|^2 |x|^\alpha dx \geq \frac{(n-2)^2 + 2\alpha(n-2)}{4} \int_{B_\rho} u(x)^2 |x|^{\alpha-2} dx + \frac{1}{4} \int_{B_\rho} u(x)^2 |x|^{\alpha-2} \Big[A_1(|x|)^{-2} + \left(A_1(|x|)A_2(|x|)\right)^{-2} + \ldots + \left(A_1(|x|)A_2(|x|)\ldots A_k(|x|)\right)^{-2}\Big] dx. \qquad (3.1)$$

holds for any $u \in W_0^{1,2}(B_\rho)$.

Remark 3.1 *If $\alpha = 0$ inequality (3.1) reduces to a known result given by inequality (2.4).*

First we introduce the following lemma needed in the proof of the main result.

Lemma 3.1 *Assume* $u \in C_0^2(B_1)$ *is radial satisfying* $u(r) > 0$ *where* $r = |x|$. *Set* $v_1(r) = u(r)r^{\frac{n-2}{2}}A_1(r)^{-\frac{1}{2}}$ *and* $v_k(r) = v_{k-1}(r)A_k(r)^{-\frac{1}{2}}$ *for* $k \geq 2$. *If* $R \geq e_k$, *then for any integer* $\eta > 0$

$$\begin{aligned}\int_{B_1} |\nabla\left(u(x)|x|^{\eta}\right)|^2 dx &= \frac{(n-2)^2-(2\eta)^2}{4}\omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r) r^{2\eta-1} dr \\ &+ \frac{\omega_n}{4}\int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r)\Big[A_1(r)^{-2} + \Big(A_1(r)A_2(r)\Big)^{-2} + \dots \\ &\qquad + \Big(A_1(r)A_2(r)\dots A_k(r)\Big)^{-2}\Big] r^{2\eta-1}dr \\ &+ \eta\omega_n \int_0^1 v_k(r)^2\Big[A_2(r)A_3(r)\dots A_k(r) + A_3(r)A_4(r)\dots A_k(r) + \dots \\ &\qquad + A_k(r) + 1\Big] r^{2\eta-1}dr \\ &+ \omega_n \int_0^1 v_k'(r)^2 A_1(r)A_2(r)\dots A_k(r) r^{2\eta+1}dr. \end{aligned} \tag{3.2}$$

for all $k \geq 1$.

Proof Since $R \geq e_k$, A_i is define for all $1 \leq i \leq k$. Let $u_\eta = u(r)r^\eta$. Then

$$u_\eta = v_k(r) r^{\frac{2-n}{2}+\eta}\left(A_1(r)A_2(r)\dots A_k(r)\right)^{\frac{1}{2}}.$$

Direct calculation gives

$$|u_\eta'|^2 = \left(\frac{n-2-2\eta}{2}\right)^2 v_k(r)^2 r^{-n+2\eta} A_1(r)A_2(r)\dots A_k(r)\Big|1 + C\Big|^2,$$

where

$$C = \frac{2}{n-2-2\eta}\left[\frac{1}{2}A_1(r)^{-1} + \dots + \frac{1}{2}\Big(A_1(r)A_2(r)\dots A_k(r)\Big)^{-1} - \frac{v_k'(r)}{v_k(r)}r\right].$$

Then

$$
\begin{aligned}
\int_{B_1} |\nabla\left(u(x)|x|^{\eta}\right)|^2 dx &= \omega_n \int_0^1 |u_\eta'|^2 r^{n-1} dr \\
&= \left(\frac{n-2-2\eta}{2}\right)^2 \omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r)\left|1+C\right|^2 r^{2\eta-1}dr \\
&= \left(\frac{n-2-2\eta}{2}\right)^2 \omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r)\left(1+2C+C^2\right) r^{2\eta-1}dr \\
&= \left(\frac{n-2-2\eta}{2}\right)^2 \omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r) r^{2\eta-1}dr \\
&\quad + (n-2-2\eta)\,\omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r)\left[\frac{1}{2}A_1(r)^{-1}+\dots\right. \\
&\qquad\qquad \left. + \frac{1}{2}\Big(A_1(r)A_2(r)\dots A_k(r)\Big)^{-1} - \frac{v_k'(r)}{v_k(r)}r\right] r^{2\eta-1}dr \\
&\quad + \omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r)\left[\frac{1}{2}A_1(r)^{-1}+\dots\right. \\
&\qquad\qquad \left. + \frac{1}{2}\Big(A_1(r)A_2(r)\dots A_k(r)\Big)^{-1} - \frac{v_k'(r)}{v_k(r)}r\right]^2 r^{2\eta-1}dr \qquad (3.3) \\
&= \left(\frac{n-2-2\eta}{2}\right)^2 \omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r) r^{2\eta-1}dr \\
&\quad - \left(\frac{n-2-2\eta}{2}\right) \omega_n \int_0^1 (v_k(r)^2)' A_1(r)A_2(r)\dots A_k(r) r^{2\eta}dr \\
&\quad + \left(\frac{n-2-2\eta}{2}\right) \omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r)\left[\frac{1}{2}A_1(r)^{-1}+\dots\right. \\
&\qquad\qquad \left. + \frac{1}{2}\Big(A_1(r)A_2(r)\dots A_k(r)\Big)^{-1}\right] r^{2\eta-1}dr \\
&\quad + \omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r)\left[\frac{1}{2}A_1(r)^{-1}+\dots\right. \\
&\qquad\qquad \left. + \frac{1}{2}\Big(A_1(r)A_2(r)\dots A_k(r)\Big)^{-1} - \frac{v_k'(r)}{v_k(r)}r\right]^2 r^{2\eta-1}dr.
\end{aligned}
$$

Applying integration by parts to second term of (3.3) we get

$$
\begin{aligned}
\int_{B_1} |\nabla\left(u(x)|x|^{\eta}\right)|^2 dx &= \frac{(n-2)^2-(2\eta)^2}{4}\omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r) r^{2\eta-1}dr \\
&\quad + \omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r)\left[\frac{1}{2}A_1(r)^{-1}+\dots\right. \\
&\qquad\qquad \left. + \frac{1}{2}\Big(A_1(r)A_2(r)\dots A_k(r)\Big)^{-1} - \frac{v_k'(r)}{v_k(r)}r\right]^2 r^{2\eta-1}dr. \qquad (3.4)
\end{aligned}
$$

Also after expanding the second term of (3.4) and by integration by parts we get (3.2). By inductive argument we will show the validity of (3.2) for all $k \geq 1$. For $k = 1$, it is easy to verify by similar calculation that

$$\begin{aligned}\int_{B_1} |\nabla (u(x)|x|^\eta)|^2 dx = &\frac{(n-2)^2-(2\eta)^2}{4}\omega_n \int_0^1 v_k(r)^2 A_1(r) r^{2\eta-1} dr \\ &+ \frac{\omega_n}{4}\int_0^1 v_k(r)^2 A_1(r)^{-1} r^{2\eta-1} dr + \eta\omega_n \int_0^1 v_k(r)^2 r^{2\eta-1} dr \\ &+ \omega_n \int_0^1 v_k'(r)^2 A_1(r) r^{2\eta+1} dr\end{aligned} \tag{3.5}$$

Since $v_{k+1}(r) = v_k(r)A_{k+1}(r)^{-\frac{1}{2}}$, direct calculation gives

$$\begin{aligned}v_k'(r)^2 =& v_{k+1}'(r)^2 A_{k+1}(r) - \frac{1}{r} v_{k+1}(r) v_{k+1}'(r)\Big(A_1(r)A_2(r)\dots A_k(r)\Big)^{-1} \\ &+ \frac{1}{4r^2} v_{k+1}(r)^2 \Big(A_1(r)A_2(r)\dots A_k(r)\Big)^{-2} A_{k+1}(r)^{-1}.\end{aligned}$$

Then the last term in the right hand side of (3.2) becomes

$$\begin{aligned}&\omega_n \int_0^1 v_k'(r)^2 A_1(r)A_2(r)\dots A_k(r) r^{2\eta+1} dr \\ &= \omega_n \int_0^1 v_{k+1}'(r)^2 A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r) r^{2\eta+1} dr + \eta\omega_n \int_0^1 v_{k+1}(r)^2 r^{2\eta-1} dr \\ &\quad + \frac{\omega_n}{4}\int_0^1 v_{k+1}(r)^2 \Big(A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r)\Big)^{-1} r^{2\eta-1} dr.\end{aligned} \tag{3.6}$$

Hence the last term in the right hand side of (3.2) generates the new terms such that

$$\begin{aligned}\int_{B_1} |\nabla (u(x)|x|^\eta)|^2 dx = &\frac{(n-2)^2-(2\eta)^2}{4}\omega_n \int_0^1 v_{k+1}(r)^2 A_1(r)A_2(r)\dots A_{k+1}(r) r^{2\eta-1} dr \\ &+ \frac{\omega_n}{4}\int_0^1 v_{k+1}(r)^2 A_1(r)A_2(r)\dots A_{k+1}(r)\Big[A_1(r)^{-2} + \Big(A_1(r)A_2(r)\Big)^{-2} + \dots \\ &\quad + \Big(A_1(r)A_2(r)\dots A_k(r)\Big)^{-2} + \Big(A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r)\Big)^{-2}\Big] r^{2\eta-1} dr \\ &+ \eta\omega_n \int_0^1 v_{k+1}(r)^2 \Big[A_2(r)A_3(r)\dots A_{k+1}(r) + A_3(r)A_4(r)\dots A_{k+1}(r) + \dots \\ &\qquad + A_k(r)A_{k+1}(r) + A_{k+1}(r) + 1\Big] r^{2\eta-1} dr \\ &+ \omega_n \int_0^1 v_{k+1}'(r)^2 A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r) r^{2\eta+1} dr\end{aligned} \tag{3.7}$$

Therefore (3.2) is valid for all $k \geq 1$. □

PROOF OF THEOREM 3.1: We shall first prove inequality (3.1) for smooth positive radially nonincreasing function defined on a unit ball B_1, centered at the origin. Then $R \geq e_k$ and for $u \in C_0^2(B_1)$, $u(r) > 0$, $r = |x|$, radially nonincreasing, we set $v_1(r) = u(r) r^{\frac{n-2}{2}} A_1(r)^{-\frac{1}{2}}$

and $v_k(r) = v_{k-1}(r)A_k(r)^{-\frac{1}{2}}$ for $k \geq 2$. Since $R \geq e_k$, A_i is define for all $1 \leq i \leq k$. Then direct calculation gives

$$\begin{aligned} u(r)u'(r) =& v_k(r)v_k'(r)r^{2-n}A_1(r)\dots A_k(r) + \frac{2-n}{2}v_k(r)^2r^{1-n}A_1(r)\dots A_k(r) \\ &- \frac{1}{2}v_k(r)^2r^{1-n}A_2(r)\dots A_k(r) - \frac{1}{2}v_k(r)^2r^{1-n}A_3(r)\dots A_k(r) - \dots \\ &- \frac{1}{2}v_k(r)^2r^{1-n}. \end{aligned} \tag{3.8}$$

Then for any integer $\eta > 0$ we have

$$\begin{aligned} 2\eta\omega_n \int_0^1 u(r)u'(r)r^{2\eta+n-2}dr =& \eta\omega_n \int_0^1 \left(v_k(r)^2\right)' A_1(r)\dots A_k(r)r^{2\eta}dr \\ &+ (2-n)\eta\omega_n \int_0^1 v_k(r)^2A_1(r)\dots A_k(r)r^{2\eta-1}dr \\ &- \eta\omega_n \int_0^1 v_k(r)^2A_2(r)\dots A_k(r)r^{2\eta-1}dr \\ &- \eta\omega_n \int_0^1 v_k(r)^2A_3(r)\dots A_k(r)r^{2\eta-1}dr - \dots \\ &- \eta\omega_n \int_0^1 v_k(r)^2r^{2\eta-1}dr \end{aligned} \tag{3.9}$$

applying integration by parts in the first term of the right hand side of (3.9) we get

$$\begin{aligned} &2\eta\omega_n \int_0^1 u(r)u'(r)r^{2\eta+n-2}dr = \\ &\qquad - \left[\eta(n-2) + 2\eta^2\right]\omega_n \int_0^1 v_k(r)^2A_1(r)\dots A_k(r)r^{2\eta-1}dr. \end{aligned} \tag{3.10}$$

Also

$$\eta^2\omega_n \int_0^1 u(r)^2r^{2\eta+n-3}dr = \eta^2\omega_n \int_0^1 v_k(r)^2A_1(r)\dots A_k(r)r^{2\eta-1}dr. \tag{3.11}$$

Hence from (3.10) and (3.11) we have

$$\begin{aligned} &2\eta\omega_n \int_0^1 u(r)u'(r)r^{2\eta+n-2}dr + \eta^2\omega_n \int_0^1 u(r)^2r^{2\eta+n-3}dr = \\ &\qquad - \left[\eta(n-2) + \eta^2\right]\omega_n \int_0^1 v_k(r)^2A_1(r)\dots A_k(r)r^{2\eta-1}dr. \end{aligned} \tag{3.12}$$

From Lemma 3.1 we have

$$\begin{aligned}
\omega_n \int_0^1 |\, (u(r)r^{\eta})' \,|^2 r^{n-1} dr = \\
&\frac{(n-2)^2-(2\eta)^2}{4}\omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r) r^{2\eta-1} dr \\
&+ \frac{\omega_n}{4}\int_0^1 v_k(r)^2 A_1(r)A_2(r)\dots A_k(r)\Big[A_1(r)^{-2} + \Big(A_1(r)A_2(r)\Big)^{-2} + \dots \\
&\qquad + \Big(A_1(r)A_2(r)\dots A_k(r)\Big)^{-2}\Big] r^{2\eta-1} dr \\
&+ \eta\omega_n \int_0^1 v_k(r)^2 \Big[A_2(r)A_3(r)\dots A_k(r) + A_3(r)A_4(r)\dots A_k(r) + \dots \\
&\qquad + A_k(r) + 1\Big] r^{2\eta-1} dr \\
&+ \omega_n \int_0^1 v_k'(r)^2 A_1(r)A_2(r)\dots A_k(r) r^{2\eta+1} dr
\end{aligned} \tag{3.13}$$

but

$$\begin{aligned}
\omega_n \int_0^1 |\, (u(r)r^{\eta})' \,|^2 r^{n-1} dr &= \omega_n \int_0^1 |u'(r)r^{\eta} + \eta u(r) r^{\eta-1}|^2 r^{n-1} dr \\
&= \omega_n \int_0^1 |u'(r)|^2 r^{2\eta+n-1} dr \\
&\quad + 2\eta\omega_n \int_0^1 u(r)u'(r) r^{2\eta+n-2} dr \\
&\quad + \eta^2\omega_n \int_0^1 u(r)^2 r^{2\eta+n-3} dr
\end{aligned}$$

hence

$$\begin{aligned}
\omega_n \int_0^1 |u'(r)|^2 r^{2\eta+n-1} dr =& \omega_n \int_0^1 |\, (u(r)r^{\eta})' \,|^2 r^{n-1} dr \\
&- 2\eta\omega_n \int_0^1 u(r)u'(r) r^{2\eta+n-2} dr \\
&- \eta^2\omega_n \int_0^1 u(r)^2 r^{2\eta+n-3} dr
\end{aligned} \tag{3.14}$$

substituting equations (3.11),(3.12) and (3.13) to equation (3.14) and by letting $\alpha = 2\eta$ we get

$$\omega_n \int_0^1 |u'(r)|^2 r^{\alpha+n-1} dr = \frac{(n-2)^2 + 2\alpha(n-2)}{4} \omega_n \int_0^1 v_k(r)^2 A_1(r) \dots A_k(r) r^{\alpha-1} dr$$
$$+ \frac{\omega_n}{4} \int_0^1 v_k(r)^2 A_1(r) A_2(r) \dots A_k(r) \Big[A_1(r)^{-2} + \Big(A_1(r) A_2(r) \Big)^{-2} + \dots$$
$$+ \Big(A_1(r) A_2(r) \dots A_k(r) \Big)^{-2} \Big] r^{\alpha-1} dr$$
$$+ \frac{\alpha\omega_n}{2} \int_0^1 v_k(r)^2 \Big[A_2(r) A_3(r) \dots A_k(r) + A_3(r) A_4(r) \dots A_k(r) + \dots$$
$$+ A_k(r) + 1 \Big] r^{\alpha-1} dr$$
$$+ \omega_n \int_0^1 v_k'(r)^2 A_1(r) A_2(r) \dots A_k(r) r^{\alpha+1} dr.$$

Since $\int_{B_1} |\nabla (u(x))|^2 |x|^\alpha dx = \omega_n \int_0^1 |u'(r)|^2 r^{\alpha+n-1} dr$ then

$$\int_{B_1} |\nabla (u(x))|^2 |x|^\alpha dx = \frac{(n-2)^2 + 2\alpha(n-2)}{4} \omega_n \int_0^1 v_k(r)^2 A_1(r) A_2(r) \dots A_k(r) r^{\alpha-1} dr$$
$$+ \frac{\omega_n}{4} \int_0^1 v_k(r)^2 A_1(r) A_2(r) \dots A_k(r) \Big[A_1(r)^{-2} + \Big(A_1(r) A_2(r) \Big)^{-2} + \dots$$
$$+ \Big(A_1(r) A_2(r) \dots A_k(r) \Big)^{-2} \Big] r^{\alpha-1} dr$$
$$+ \frac{\alpha\omega_n}{2} \int_0^1 v_k(r)^2 \Big[A_2(r) A_3(r) \dots A_k(r) + A_3(r) A_4(r) \dots A_k(r) + \dots \qquad (3.15)$$
$$+ A_k(r) + 1 \Big] r^{\alpha-1} dr$$
$$+ \omega_n \int_0^1 v_k'(r)^2 A_1(r) A_2(r) \dots A_k(r) r^{\alpha+1} dr$$

By inductive argument we will show the validity of (3.15) for all $k \geq 1$. For $k = 1$, it is easy to verify by similar calculation that

$$\int_{B_1} |\nabla (u(x))|^2 |x|^\alpha dx = \frac{(n-2)^2 + 2\alpha(n-2)}{4} \omega_n \int_0^1 v_k(r)^2 A_1(r) r^{\alpha-1} dr$$
$$+ \frac{\omega_n}{4} \int_0^1 v_k(r)^2 A_1(r)^{-1} r^{\alpha-1} dr + \frac{\alpha\omega_n}{2} \int_0^1 v_k(r)^2 r^{\alpha-1} dr$$
$$+ \omega_n \int_0^1 v_k'(r)^2 A_1(r) r^{\alpha+1} dr.$$

Since $v_{k+1}(r) = v_k(r)A_{k+1}(r)^{-\frac{1}{2}}$, direct calculation gives

$$\begin{aligned} v_k'(r)^2 =& v_{k+1}'(r)^2 A_{k+1}(r) - \frac{1}{r} v_{k+1}(r) v_{k+1}'(r) \Big(A_1(r)A_2(r)\dots A_k(r) \Big)^{-1} \\ &+ \frac{1}{4r^2} v_{k+1}(r)^2 \Big(A_1(r)A_2(r)\dots A_k(r) \Big)^{-2} A_{k+1}(r)^{-1}. \end{aligned}$$

Then the last term in the right hand side of (3.15) becomes

$$\begin{aligned} &\omega_n \int_0^1 v_k'(r)^2 A_1(r)A_2(r)\dots A_k(r) r^{\alpha+1} dr \\ &= \omega_n \int_0^1 v_{k+1}'(r)^2 A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r) r^{\alpha+1} dr + \frac{\alpha\omega_n}{2} \int_0^1 v_{k+1}(r)^2 r^{\alpha-1} dr \\ &\quad + \frac{\omega_n}{4} \int_0^1 v_{k+1}(r)^2 \Big(A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r) \Big)^{-1} r^{\alpha-1} dr. \end{aligned}$$

Hence the last term in the right hand side of (3.15) generates the new terms such that

$$\begin{aligned} \int_{B_1} |\nabla (u(x))|^2 |x|^\alpha dx =& \frac{(n-2)^2 + 2\alpha(n-2)}{4} \omega_n \int_0^1 v_{k+1}(r)^2 A_1(r)A_2(r)\dots A_{k+1}(r) r^{\alpha-1} dr \\ &+ \frac{\omega_n}{4} \int_0^1 v_{k+1}(r)^2 A_1(r)A_2(r)\dots A_{k+1}(r) \Big[A_1(r)^{-2} + \Big(A_1(r)A_2(r) \Big)^{-2} + \dots \\ &\quad + \Big(A_1(r)A_2(r)\dots A_k(r) \Big)^{-2} + \Big(A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r) \Big)^{-2} \Big] r^{\alpha-1} dr \\ &+ \frac{\alpha\omega_n}{2} \int_0^1 v_{k+1}(r)^2 \Big[A_2(r)A_3(r)\dots A_{k+1}(r) + A_3(r)A_4(r)\dots A_{k+1}(r) + \dots \\ &\quad + A_k(r)A_{k+1}(r) + A_{k+1}(r) + 1 \Big] r^{\alpha-1} dr \\ &+ \omega_n \int_0^1 v_{k+1}'(r)^2 A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r) r^{\alpha+1} dr \end{aligned}$$

Therefore (3.15) is valid for all $k \geq 1$.

Ignoring the last two terms in the right hand side of (3.15) and for

$$v_k(r)^2 = u(r)^2 r^{n-2} \Big(A_1(r)A_2(r)\dots A_k(r) \Big)^{-1}, \quad k \geq 1$$

we get

$$\begin{aligned} \int_{B_1} |\nabla u(x)|^2 |x|^\alpha dx \geq& \frac{(n-2)^2 + 2\alpha(n-2)}{4} \int_{B_1} u(x)^2 |x|^{\alpha-2} dx \\ &+ \frac{1}{4} \int_{B_1} u(x)^2 |x|^{\alpha-2} \Big[A_1(|x|)^{-2} + \Big(A_1(|x|)A_2(|x|) \Big)^{-2} + \dots \\ &\quad + \Big(A_1(|x|)A_2(|x|)\dots A_k(|x|) \Big)^{-2} \Big] dx. \end{aligned} \tag{3.16}$$

Hence inequality (3.1) holds for domain B_1. By density argument, inequality (3.1) is valid for any $u \in W_0^{1,2}(B_\rho)$, $u \geq 0$. Thus theorem 3.1 follows. □

References

[1] **Adimurthi, Chaudhuri, N., and Ramaswamy, M.,** (2001). *An improved Hardy-Sobolev inequality and its application.* Proceedings of the American Mathematical Society. **Vol. 130**, No. 2, pp 489-505.

[2] **Detalla, A. L., Horiuchi, T., and Ando, H.,** (2004). *Missing Terms in Hardy-Sobolev Inequality.* Proceedings of the Japan Academy. **Vol. 80,** Ser. A, No. 8, pp 160-165.

[3] **Detalla, A. L., Horiuchi, T., and Ando, H.,** (2004). *Missing Terms in Hardy-Sobolev Inequality and its Application.* Far East Journal of Mathematical Sciences. **Vol. 14,**, No. 3, pp 333-359.

[4] **Detalla, A. L., Horiuchi, T., and Ando, H.,** (2005). *Sharp Remainder Terms of Hardy-Sobolev Inequalities.* Mathematical Journal of Ibaraki University. **Vol. 37,** pp 39-52.

[5] **Detalla, A. L., Horiuchi, T., and Ando, H.,** (2005). *Sharp Remainder Terms of Hardy-Sobolev Inequalities.* Preprint 2005:32, Department of Mathematical Sciences, Division of Mathematics, Chalmers University of Technology/Goteborg University.