Volume 3 Issue 1 May 2012

The MINDANAWAN

Journal of Mathematics

ISSN 2094-7380

Resistance Distance in Complete n-Partite Graphs
Severino V. Gervacio 1

Integrable Set and Measurable Function
Julius V. Benitez 20

An Integer Programming Approach for Optimizing Street Food Snack Nutrient Intake Among College Students
Rhoda P. Agdeppa-Namoco 28

On the AB-Generalized Fibonacci, Pell and Jacobsthal Sequences by Hessenberg Matrices
Mhelmar A. Labendia, Michael B. Frondoza 41

Another Look at Geodetic and Hull Numbers of a Graph
Dennis Tarepe, Sergio R. Canoy, Jr. 52

Intersection Number of Some Graphs
Jesrael B. Palco, Rolando N. Paluga 63

Bridging the Gap: Remediating Mathematics Competencies of College Entrants
Leorence C. Tandog 76

A Proposed Two-Stage Sequential Point Estimation of the Hazard Rate Function
Aljo Clair P. Pingal, Daisy Lou L. Polestico 85

McShane Integral in Locally Convex Topological Vector Spaces
Juniven Acapulco, Julius V. Benitez 91

Resistance Distance in Complete n-Partite Graphs

Severino V. Gervacio
De La Salle University, 2401 Taft Avenue, 1004 Manila, Philippines
emal@mail.com

Abstract: We may view any graph as a network of resistors each having a resistance of 1 ohm. The *resistance distance* between a pair of vertices in a graph is defined as the effective resistance between the two vertices. This function is known to be a metric on the vertex-set of any graph. The main result of this paper is an explicit expression for the resistance distance between any pair of vertices in the complete n-partite graph $K_{m_1,m_2,\ldots,m_n}$.

1 Introduction

We shall deal here with finite graphs without loops but possibly with multiple edges. The vertex-set of a graph G is denoted by $V(G)$ and its edge-set is denoted by $E(G)$.

The usual *distance* from a vertex x to a vertex y in a graph G is defined to be the length of any shortest path joining x to y. Thus $d\colon V(G) \times V(G) \to \mathbb{R}$ is a function that satisfies

1. $d(x, y) \geq 0$ for all $x, y \in V(G)$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$ for all $x, y \in V(G)$
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in V(G)$

If A is any set, any function μ from $A \times A$ to the set $\mathbb{R}$ of real numbers satisfying the four conditions above is called a *metric* on A.

Here we shall deal with a certain metric on the vertex-set of a graph G called *resistance distance*.

We associate a graph G with a network $N(G)$ of unit resistors (resistor with resistance 1 ohm) in the most natural way—each edge of G is a unit resistor in $N(G)$. For example, the fan F_3 and the associated network $N(F_3)$ are shown in Figure 1.

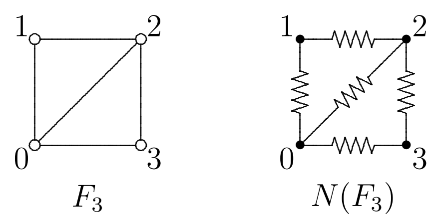


Figure 1: The fan F_3 and the associated network of unit resistors $N(F_3)$

If a source of electromotive force is connected to two nodes of the network, say at 0 and 1, current will flow into and out of the network. According to Ohm's law, if the difference in potential between two nodes is V and the current that flows into one node and out of another node is I, then $V = IR$, where R is the effective resistance between the two nodes. Please refer to Figure 2.

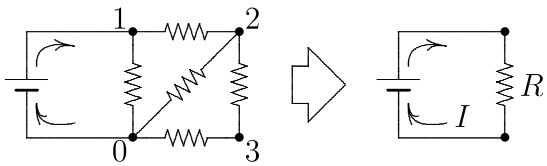


Figure 2: An electrical circuit with effective resistance R between 0 and 1

Ohm's law easily yield formulas for the effective resistance of *resistors in series* or *resistors in parallel.*

Figure 3 shows n resistors connected in series.

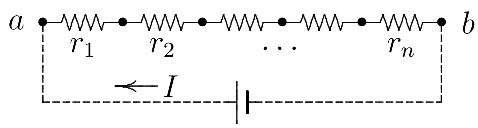


Figure 3: Resistors in series with respect to a and b

According to Ohm's law, the current that flows through a resistor in an electrical circuit is equal to the potential difference between the terminals of the resistor divided by the resistance of the resistor. If V is the potential difference, i is the current, and R is the resistance, then $i = V/R$. Here, resistor may be replaced by a network of resistors.

If n resistors are in series, the current that flows through each of them is the same. Referring to Figure 3, let the potential at a be V_a and that at b be V_b. Then the potential difference between a and b is $V = V_a - V_b$ if we assume that the potential at a is higher. Potential is much like pressure. We can compare the situation to water flowing in a pipe through loads $r_1, \ldots, r_n$. There are different pressures at points between adjacent loads. The summation of all pressure differences from a to b is equal to the difference between the pressures at a and b. Going back to the original set-up of resistors, the potential difference between the terminals of r_1 is ir_1. The summation of all potential differences is

$$i(r_1 + r_2 + \cdots + r_n)$$

If we view the resistors in series as one single resistor with resistance R, then the potential difference between a and b is iR. Therefore, we get the equation

$$iR = i(r_1 + r_2 + \cdots + r_n)$$

Hence, if n resistors with resistances of r_1, r_2, $\ldots$, r_n ohms are in series, their effective resistance is

$$\Omega(a, b) = r_1 + r_2 + \cdots + r_n \tag{1}$$

Figure 4 shows n resistors in parallel. Assume that the total current entering at a is j and that the current flowing through r_i is j_i. Then $j = j_1 + j_2 + \cdots + j_n$.

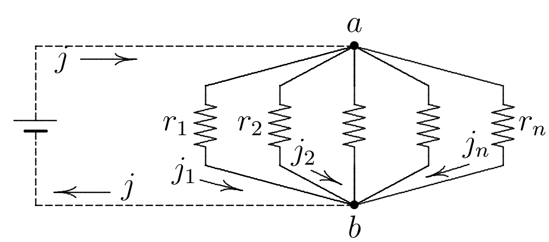


Figure 4: Resistors in parallel with respect to a and b

If we denote by R the effective resistance between a and b, then $j = V/R$, where V is the potential difference between a and b. But a and b are common terminals of all the resistors. Therefore, we have

$$j = j_1 + j_2 + \cdots + j_n$$
$$\frac{V}{R} = \frac{V}{r_1} + \frac{V}{r_2} + \cdots + \frac{V}{r_n}$$

Therefore, if n resistors with resistances of $r_1, r_2, \ldots, r_n$ ohms are in parallel, their effective resistance $\Omega(a,b)$ satisfies

$$\frac{1}{\Omega(a,b)} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n} \tag{2}$$

Some networks of resistors may be simplified to a single resistor using series and parallel connections analysis. Consider the network shown in Figure 5. The effective resistance between a and b is computed using equations (1) and (2). In each step of the simplification, the equation used is indicated in the block arrow.

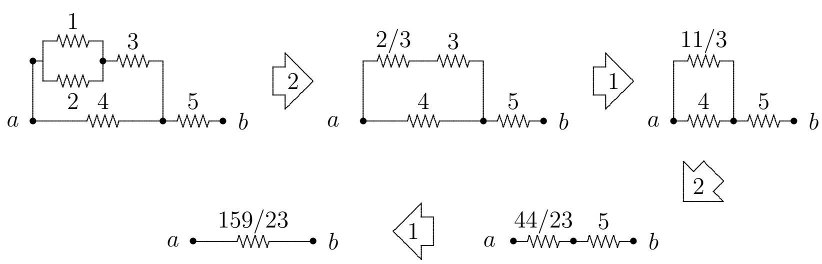


Figure 5: Simplification *via* series and parallel connections

Unfortunately, the method of simplification illustrated in this example may not work for some pairs of vertices in a network of resistors. Consider the network N shown in Figure 6. The following effective resistances are easy to get:

$$\Omega(a,c) = \frac{17}{21}, \Omega(a,d) = \frac{26}{21}, \Omega(b,c) = \frac{52}{21},$$
$$\Omega(b,d) = \frac{55}{21}, \Omega(c,d) = \frac{9}{7}$$

The computation of $\Omega(a,b)$ cannot be done directly using series and parallel connections analysis.

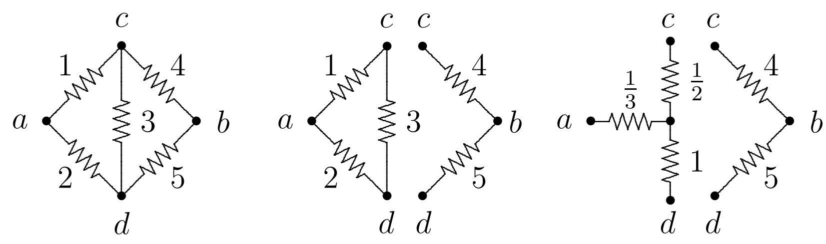


Figure 6: Network decomposition and substitution

If we look at the triangle acd in the middle part of Figure 6, the effective resistance between a and d is computed by noting that the resistances of 1 and 3 are in series (with respect to a and d) and their sum, 4, is in parallel with the resistance of 2. Therefore, $\Omega(a,d) = 4/3$. Likewise, $\Omega(a,c) = 5/6$ and $\Omega(c,d) = 3/2$.

We are substituting for the triangle acd the star with three resistors shown on the right of Figure 6. In this star, the effective resistance between a and d is that of two resistors in series, that is $1 + \frac{1}{3} = 4/3$. This is so because the resistance of $1/2$ does not affect the unique path joining a and d. We are therefore *eliminating* the resistor with one terminal at d. We see that the effective resistance between any pairs of vertices in the triangle is the same as that in the star.

Going back to the computation of effective resistance between a and b for the network N in Figure 6, the details are shown in Figure 7.

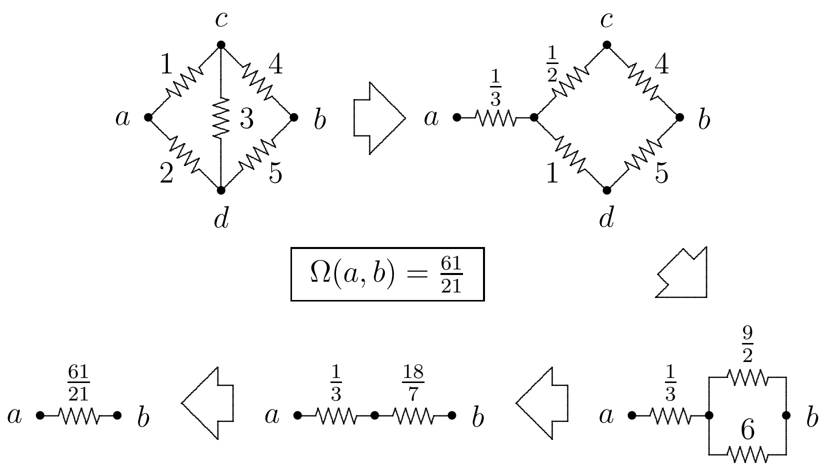


Figure 7: Simplification by substitution

For a graph G, Klein and Randić defined a function $\Omega\colon V(G) \times V(G) \to \mathbb{R}$ called *resistance distance* [6].

Definition 1.1 Let G be a graph. The *resistance distance* between two vertices i and j, denoted by $\Omega(i, j)$, is the effective resistance between i and j in the network of unit resistors $N(G)$.

Remark 1.2 We sometimes write $\Omega_G(i, j)$ to express the fact that i and j are vertices of the graph G. It is known that the function Ω is a metric on the vertex-set of a graph.

The main contribution of this paper is an explicit expression for the resistance distance between vertices in complete n-partite graphs.

2 Some Known Results

Klein [7] determined the resistance distances in the graphs of the five Platonic solids. Bapat and Gupta [1] gave interesting formulas in terms of Fibonacci numbers for the resistance distances in wheels and fans. In [4] the limit of the resistance distance between some pairs of vertices in fans and wheels are determined as the order of the graph increases without bound.

The *Laplacian matrix* of a graph with vertices $1, 2, 3, \ldots, n$ is the matrix $L(G) = [\ell_{i,j}]$ where

$$\ell_{i,j} = \begin{cases} \deg(i) & \text{if } i = j, \\ -1 & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0 & \text{if } i \text{ and } j \text{ are non-adjacent.} \end{cases}$$

Equivalently, $L(G) = D(G) - A(G)$ where $D(G)$ is the diagonal matrix with diagonal entries $\deg(1), \deg(2), \ldots, \deg(n)$ and $A(G)$ is the adjacency matrix of G.

Bapat and Gutman [2] gave a simple formula for resistance distance in terms of determinants of minors of $L(G)$.

Klein [7] showed that for a connected graph, the summation of effective resistances between all pairs of adjacent vertices is equal to the total number of vertices minus 1. This result has actually been determined a long time ago by Foster [3] and Weinberg [8].

Theorem 2.1 (Klein, [7]) *Let G be a connected graph of order n, Then $\sum \Omega(i, j) = n-1$ where the summation ranges over all pairs of adjacent vertices i and j.*

3 Zero and Negative Resistances

In reality, resistors have positive resistances. However, we will find it useful to introduce the notion of zero and negative resistances. At the same time we extend equations (1) and (2) to include zero and negative resistances.

If a resistor with zero resistance is in series with other resistors, then it does not affect the effective resistance of the resistors in series in view of equation (1). If a resistor with zero resistance is in parallel with some resistors, then the effecticve resistance of the parallel resistors becomes 0 in view of equation (2).

If a resistor with a resistance of r ohms is in parallel with a resistor with a resistance of $-r$ ohms, then their effective resistance is $+\infty$. This means that the nodes to which the two resistors are connected become effectively disconnected.

It should be noted however that with the introduction of non-positive values of resistance, the function Ω is no longer a metric.

4 Two Useful Principles in Electrical Circuits

In a given network, we use the notation $\Omega(i,j)$ to denote the effective resistance between two vertices i and j. In case i and j are adjacent, we use the notation $\omega(i,j)$ for the resistance of the resistor with terminals at i and j.

From hereon, we allow resistors to have any real value. To facilitate the proof of our main result, we first discuss two important principles in electrical circuits, particularly network of resistors.

4.1 Principle of Elimnation

Recall that a *cut-vertex* of a connected graph G is a vertex whose removal from G disconnects G. A *block* of G is a maximal connected subgraph of G which does not contain a cut-vertex of itself.

Let N be a network of resistors with underlying graph G which is connected. Let B be a block of G containing exactly one cut-vertex x of G. If N' is the network obtained from N by deleting all the vertices of B except x, then for all i, j in N', $\Omega_N(i,j) = \Omega_{N'}(i,j)$. This is the *principle of elimination* [7].

As an example, in Figure 8, the effective resistance between the vertices a and b is $\Omega(a,b) = 2 + 3 = 5$.

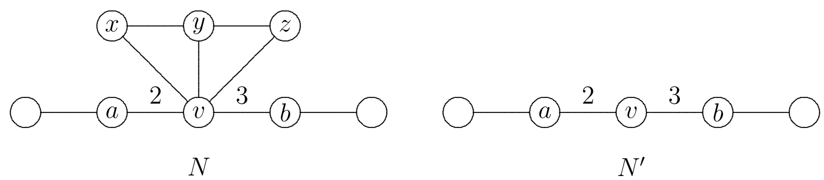


Figure 8: Eliminating the edges of a block from a network

The subgraph induced by the vertices x, y, z, v is a block of N and if we remove the vertices x, y, z from N we get the network N' shown in Figure 8. Applying the principle of elimination to N' two more times, we are left with a network with two resistors of 2 and 3 ohms in series.

4.2 Principle of Substitution

If N is a network of resistors, we may look at N as a *weighted graph* where the weight of an edge is the resistance of the resistor represented by the edge. For convenience, we introduce the concept of S-equivalent networks.

Definition 4.1 Let N and M be networks of resistors and let $S \subseteq V(N) \cap V(M)$. we say that N and M are S-*equivalent* if $\Omega_N(i,j) = \Omega_M(i,j)$ for all $i, j \in S$.

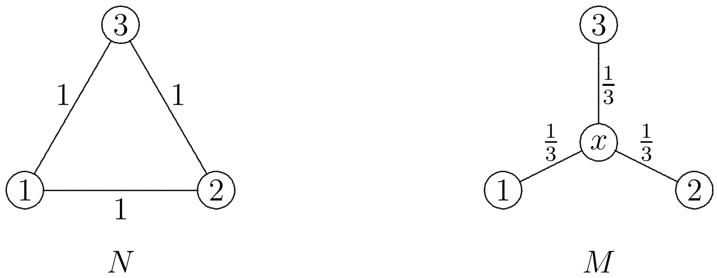
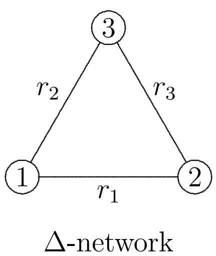


Figure 9: $\{1, 2, 3\}$-equivalent networks

The two networks N and M shown in Figure 9 are $\{1, 2, 3\}$-equivalent. The network N is called a Δ-network while the network M is called a Y-network. A Δ-network is convertible to an equivalent Y-network by the formulas indicated in Figure 10. These formulas were first derived by Kennelly [5] in 1899.



$$R_1 = \frac{r_2 r_3}{r_1 + r_2 + r_3}$$
$$R_2 = \frac{r_1 r_3}{r_1 + r_2 + r_3}$$
$$R_3 = \frac{r_1 r_2}{r_1 + r_2 + r_3}$$

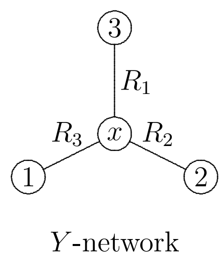


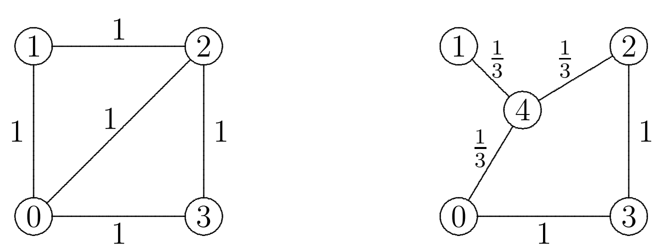
Figure 10: Transformation from Δ to Y

The formulas in Figure 10 can easily be easily derived using the fact that between any two vertices in a Δ-network, we have a resistor in parallel with a pair of resistors in series. On the other hand, in a Y-network, the effective resistance between two vertices is the sum of two resistances in series. We simply require that for all i, j, we must have $\Omega\Delta(i, j) = \Omega_Y(i, j)$. So the formulas in Figure 10 are obtained by solving the following system of equations:

$$\Omega_Y(2,3) = R_1 + R_2 = \left(\frac{1}{r_3} + \frac{1}{r_1 + r_2}\right)^{-1} = \Omega_\Delta(2,3)$$
$$\Omega_Y(2,1) = R_2 + R_3 = \left(\frac{1}{r_1} + \frac{1}{r_2 + r_3}\right)^{-1} = \Omega_\Delta(2,1)$$
$$\Omega_Y(1,3) = R_3 + R_1 = \left(\frac{1}{r_2} + \frac{1}{r_1 + r_3}\right)^{-1} = \Omega_\Delta(1,3)$$

The *principle of substitution* states that if H is a subnetwork of N and H is $V(H)$-equivalent to H^*, then the network N^* obtained from N by replacing H by H^* satisfies $\Omega_N(i,j) = \Omega_{N^*}(i,j)$ for all $i,j \in V(N)$, *i.e.*, N is $V(N)$-equivalent to N^*.

If G is a graph, then for convenience G will also denote the network $N(G)$. To illustrate the principle of substitution, consider the fan F_3 in Figure 1. This network is $\{0,1,2,3\}$-equivalent to the network F_3^* shown in Figure 11.



F_3 $\qquad$ F_3^*

Figure 11: $V(F_3)$-equivalent networks

Consider the complete bipartite graph $K_{m,n} = \overline{K}_m + \overline{K}_n$. If $i \in V(\overline{K}_m$ and $i \in V(\overline{K}_n)$, then $\Omega(i,j)$ is independent of i and j. Since $K_{m,n}$ has mn edges, then $mn\Omega(i,j) = m+n-1$

by Theorem 2.1. Thus, $\Omega(i,j) = \dfrac{m+n-1}{mn}$ if i and j are adjacent vertices.

Theorem 4.2 (Klein, [7]) *Let i, j be distinct vertices in the complete bipartite graph $K_{m,n}$. Then*

$$\Omega(i,j) = \begin{cases} \frac{2}{n} & \text{if } i,j \in V(\overline{K}_m), \\ \frac{2}{m} & \text{if } i,j \in V(\overline{K}_n), \\ \frac{m+n-1}{mn} & \text{if } i \in V(K_m) \text{ and } i \in V(K_n). \end{cases}$$

Let us define the network $K^*_{m,n}$ to be the network whose underlying graph consists of $m+n+2$ vertices x_i, $0 \le i \le m$ and y_i, $0 \le i \le n$ with edges $[x_0, y_0]$; $[x_0, x_i]$, $1 \le i \le m$; and $[y_0, y_i]$, $1 \le i \le n$. See Figure 12.

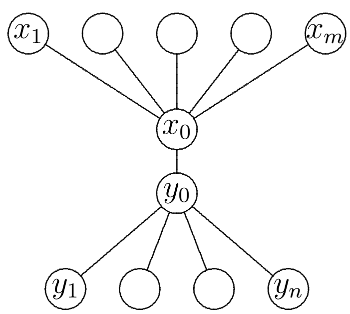


Figure 12: The underlying graph of the network $K^*_{m,n}$

The resistances of the edges of $K^*_{m,n}$ are $\omega(x_0, x_i) = \frac{1}{n}$ for $i \ne j$, $\omega(y_0, y_i) = \frac{1}{m}$ for $i \ne 0$, and $\omega(x_0, y_0) = -\frac{1}{mn}$.

By the principle of elimination, we see that $\Omega(x_i, x_j) = \frac{2}{n}$ if $i \ne j$; $\Omega(y_i, y_j) = \frac{2}{m}$ if $i \ne j$,; $\Omega(x_i, y_j) = \frac{1}{n} - \frac{1}{mn} + \frac{1}{m} = \frac{m+n-1}{mn}$ for all i and j. Therefore, we have established the following theorem.

Theorem 4.3 *The network $K_{m,n}$ is $V(K_{m,n})$-equivalent to $K^*_{m,n}$.*

Corollary 4.4 *Let G be a graph with two non-adjacent vertices a and b such that $N(a) = N(b)$. Then* $\Omega(a,b) = \dfrac{2}{|N(a)|}$.

Proof: To prove this Corollary, we apply the principles of substitution and elimination. We replace the subgraph $K_{2,n}$, where $n = |N(a)|$, by the network $K^*_{2,n}$. The conclusion then follows after applying the principle of elimination. □

5 Complete n-Partite Graph

The complete n-partite graph is denoted by $K_{m_1,m_2,\ldots,m_n}$. This graph is the disjoint union of $\overline{K}_{m_i}$, $i = 1, 2, \ldots, n$, *i.e.*,

$$K_{m_1,m_2,\ldots,m_n} = \overline{K}_{m_1} + \overline{K}_{m_2} + \cdots + \overline{K}_{m_n}.$$

Note that $K_{m_1,m_2,\ldots,m_n}$ is the edge-disjoint union of K_{m_1,m_2} and $K_{m_1+m_2,m_3,\ldots,m_n}$. Now K_{m_1,m_2} is $V(K_{m_1,m_2})$-equivalent to $K^*_{m_1,m_2}$ shown in Figure 13.

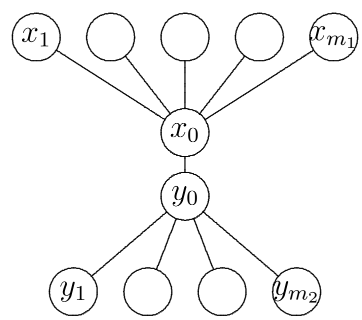
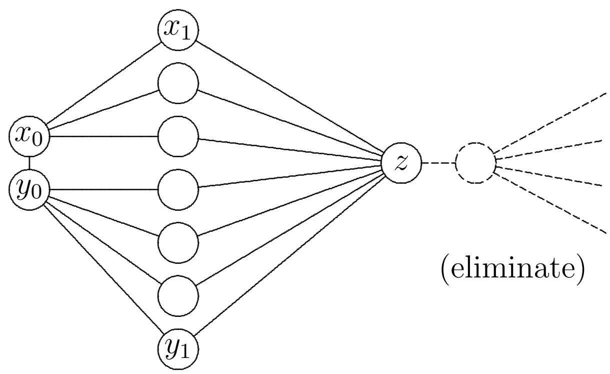


Figure 13: The underlying graph of the network $K^*_{m_1,m_2}$

On the other hand, $K_{m_1+m_2,m_3,\ldots,m_n}$ contains the spanning subgraph $K_{m_1+m_2,m_{1,2}}$ where $m_{1,2} = \sum_{i=1}^{n} m_i - (m_1+m_2)$.

We substitute in place of this spanning subgraph its equivalent $K^*_{m_1+m_2,m_{1,2}}$. Then apply the principle of elimination to obtain the graph in Figure 14. We need this graph only to get $\Omega(x_1, y_1)$.



The network N

Figure 14: Determining the effective resistance between x_1 and y_1 in $K_{m_1,m_2,\ldots,m_n}$

Figure 14 is the result after replacing the subgraph $K_{m_1+m_2.m_{1,2}}$ by the equivalent network $K^*_{m_1+m_2,m_{1,2}}$. Here we have the following values of resistances:

$$\begin{aligned}
\omega(x_0, y_0) &= -\frac{1}{m_1 m_2}, \\
\omega(x_0, x_i) &= \frac{1}{m_2} \text{ for } i \neq 0, \\
\omega(y_0, y_i) &= \frac{1}{m_1} \text{ for } i \neq 0, \\
\omega(z, x_i) &= \frac{1}{m_{1,2}} \text{ for all } i \\
\omega(z, y_j) &= \frac{1}{m_{1,2}} \text{ for all } j.
\end{aligned}$$

The network N in Figure 14 is $\{x_0, y_0, x_1, y_1, z\}$-equivalent to the network N' shown in Figure 15 (a). We obtain N' by simplifying some series-parallel connections to their equivalent.

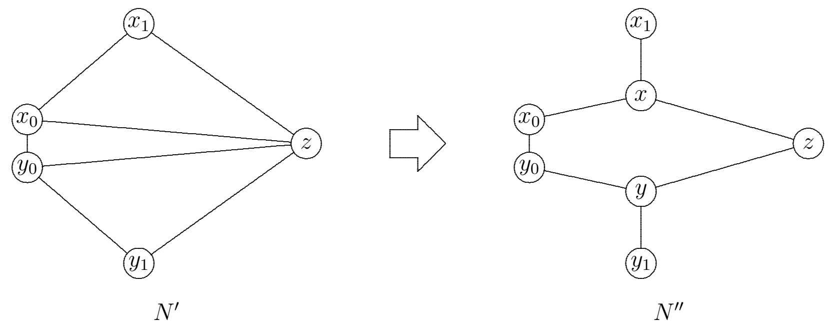


Figure 15: The network N' subjected to Δ-to-Y substitutions

In what follows, we use the fact that k resistors in parallel, each having a resistance of r, has effective resistance of $\frac{r}{k}$. In the network N', we have the following resistances between pairs of adjacent vertices:

$$\omega(x_0, y_0) = -\frac{1}{m_1 m_2} \quad , \ \omega(z, x_1) = \frac{1}{m_{1,2}}$$
$$\omega(x_0, x_1) = \frac{1}{m_2} \quad , \ \omega(z, y_1) = \frac{1}{m_{1,2}}$$
$$\omega(y_0, y_1) = \frac{1}{m_1} \quad , \ \omega(z, x_0) = \frac{1}{m_1 - 1}\left(\frac{1}{m_2} + \frac{1}{m_{1,2}}\right)$$
$$\omega(z, y_0) = \frac{1}{m_2 - 1}\left(\frac{1}{m_1} + \frac{1}{m_{1,2}}\right)$$

Our next step is to replace each of the Δ-networks $x_0 z x_1$ and $y_0 z y_1$ by their equivalent Y-networks. Please refer to

Figure 15 for the resulting network N'' after two Δ-Y transformations.

Referring to the network N'' in Figure 15, by Δ to Y conversion, we have

$$
\begin{aligned}
\omega(x_0, y_0) &= -\frac{1}{m_1 m_2} &&, \ \omega(y, y_0) = \frac{1}{m_1 m_2} \\
\omega(x, x_0) &= \frac{1}{m_1 m_2} &&, \ \omega(y, y_1) = \frac{m_2 - 1}{m_2(m_1 + m_2)} \\
\omega(x, x_1) &= \frac{m_1 - 1}{m_1(m_1 + m_2)} &&, \ \omega(y, z) = \frac{1}{m_2 m_{1,2}} \\
\omega(x, z) &= \frac{1}{m_1 m_{1,2}}
\end{aligned}
$$

Between nodes x_1 and y_1 in the network N'' we have a combination of resistors in series and parallel. Therefore by equations (1) and (2), we have

$$
\begin{aligned}
\Omega(x_1, y_1) &= \omega(x, x_1) + \omega(y, y_1) \\
&\quad + \frac{1}{\frac{1}{\omega(x,z) + \omega(y,z)} + \frac{1}{\omega(x, x_0) + \omega(y, y_0) + \omega(x_0, y_0)}} \\
&= \frac{m_1 - 1}{m_1(m_2 + m_{1,2})} + \frac{m_2 - 1}{m_2(m_1 + m_{1,2})} \\
&\qquad + \frac{1}{\left(\frac{1}{m_1 m_{1,2}} + \frac{1}{m_2 m_{1,2}}\right)^{-1} + \left(\frac{1}{m_1 m_2}\right)^{-1}} \\
&= \frac{(p-1)(2p - m_1 - m_2)}{p(p - m_1)(p - m_2)}
\end{aligned}
$$

where $p = m_1 + m_2 + \cdots + m_n$.

In general, if $a \in V(\overline{K}_{m_i})$ and $b \in V(\overline{K}_{m_j})$, then

$$\Omega(a,b) = \frac{(p-1)(2p-m_i-m_j)}{p(p-m_i)(p-m_j)}$$

We summarize our results in the next theorem.

Theorem 5.1 *Let $n > 0$ and $m_i > 0$ be integers and $p = m_1 + m_2 + \cdots + m_n$. If a and b are distinct vertices in $K_{m_1,m_2,\ldots,m_n} = \overline{K}_{m_1} + \overline{K}_{m_2} + \cdots + \overline{K}_{m_m}$ then*

$$\Omega(a,b) = \begin{cases} \dfrac{2}{p-m_i} & \text{if } a,b \in V(\overline{K}_{m_i}), \\ \dfrac{(p-1)(2p-m_i-m_j)}{p(p-m_i)(p-m_j)} & \text{if } a \in V(\overline{K}_{m_i}), \\ & b \in V(\overline{K}_{m_j}) \text{ and } i \neq j. \end{cases}$$

Remark 5.2 In case a and b belong to different sets $V(\overline{K}_{m_i})$ and $V(\overline{K}_{m_j})$, it might be easier to remember the formula for resistance distance in the following form:

$$\Omega(a,b) = \frac{p-1}{p}\left(\frac{1}{p-m_i} + \frac{1}{p-m_j}\right)$$

References

[1] R. B. Bapat and S. Gupta, (2009), *Resistance Distance in Wheels and Fans*, preprint.

[2] R. B. Bapat, I. Gutman, and W. J. Xiao, *A Simple Method for Computing Resistance Distance*, Z. Naturforsch, (2003), **58a** : 494–498

[3] R. M. Foster, in: *Contrbutions to Applied Mechanics*, Edward Brothers, Ann Arbor, Michigan, (1949) 333–340.

[4] S. V. Gervacio, *Resistance Distance in Graphs*, in: Proceedings of the 11th Osaka University-De La Salle University Academic Research Workshop, (2008) 6–9

[5] A. E. Kennelly, *Equivalence of triangles and stars in conducting networks*, Electrical World and Engineer, vol. 34, (1899) 413–414.

[6] D. J. Klein and M. Randić, *Resistance Distance*, Journal of Mathematical Chemistry **12** (1993) 81–95.

[7] D. J. Klein, *Resistance-Distance Sum Rules*, Croatica Chemica Acta, (2002), **75(2)**: 633–649

[8] L. Weinberg, *IRE Trans. Circuits Theory* **5** (1958) 8–30.