TMJM
THE MINDANAWAN
Journal of Mathematics

The Mindanawan Journal of Mathematics
Official Journal of the Department of Mathematics and Statistics
Mindanao State University-Iligan Institute of Technology
ISSN: 2094-7380 (Print) | 2783-0136 (Online)

Vol. 6 (2024), no. 1, pp. 11–27

On Henstock Approach to Uncertain Integral with respect to a Liu Process

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Received: 31st March 2024 Revised: 22nd July 2024

Abstract

In this paper, we introduce a new Henstock-type integral for the uncertain process with respect to a Liu process called Liu-Henstock integral. We show that the Liu-Henstock integral adheres to the standard properties of an integral.

1 Introduction

Over three centuries ago, it is undeniable that the world has benefited from the practical applications of integration, particularly the Riemann integration, a well-known integral learned in elementary calculus. However, it turns out that the said integral has several principal defects. The most serious one is that the class of Riemann integrable functions is restricted.

In 1902, another approach to integration arose called the Lebesgue integral, formulated by Henri Lebesgue to address the defects of the Riemann integral. Even though it can handle a broader class of functions than the Riemann integral, it still possesses certain limitations. In addition, it also demands an extensive study of measure theory, which can be challenging, especially for non-mathematicians.

In the 1960s, the Henstock-Kurzweil, or HK integral, was studied independently by Ralph Henstock and Jaroslav Kurzweil. In this study, however, we will refer to this integral simply as Henstock integral. In some sense, the Henstock integral is more general than the Riemann and Lebesgue integrals (see [8]). Since then, Henstock integration has been profoundly studied by numerous researchers (see [3, 4, 5, 7, 9]). Contrary to the Riemann integral, the Henstock integral employed non-uniform meshes. Such a technique is now known as the Henstock approach.

In stochastic calculus, the utilization of the Riemann approach to define the stochastic integral is unattainable because the integrators exhibit paths with unbounded variation, and the integrands display so many oscillations. What causes this issue is the use of uniform meshes in the Riemann sums (see [18]). Moreover, the classical way of defining the stochastic integral closely resembles that of the Lebesgue integral for a measurable function. Consequently, several

*Corresponding author
2020 Mathematics Subject Classification: 26A39, 60H05
Keywords and Phrases: Liu-Henstock integral, Liu integral, Uncertain process, Liu process, Uncertainty theory
This research has been supported by the Department of Science and Technology-Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP).

authors have studied and explored the Henstock approach to stochastic integration to provide a way out of this alleged deadlock and lessen the technicalities when defining the stochastic integral (see [6, 10, 16, 17]).

It is well-established that stochastic processes are described based on the probability theory. Using the probability theory necessitates a large sample size to estimate the probability distribution based on long-run frequency. Liu noted in [15], however, that in reality, the sample size is frequently too small or sometimes non-existent. It appears that inviting some domain experts is needed to evaluate their belief degree about the event occurring. Due to the tendency of human beings to overweight unlikely events [19], the belief degree may have a larger variance than the long-run frequency. These facts motivated Liu [13] to discover an uncertainty theory, a branch of mathematics dealing with human uncertainty (see [2]).

Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A number $\mathcal{M}(\Lambda)$ indicates the level that each $\Lambda \in \mathcal{L}$ (which is called an event) will occur. Then a set function $\mathcal{M}$ from $\mathcal{L}$ to $[0,1]$ is called an uncertain measure if it satifies the following three axioms:

Axiom 1. (Normality) $\mathcal{M}(\Gamma) = 1$.

Axiom 2. (Self-Duality) $\mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1$ for any event Λ.

Axiom 3. (Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

The probability measure met the conditions to be an uncertain measure. However, the converse is not true as the uncertain measure failed to conform to the countable additivity property, making the uncertainty space more general than the probability space. Despite this, probability theory is not a special case of uncertainty theory since the uncertainty theory assumes product uncertain measure is the minimum of uncertain measures of individual events, while probability theory assumes product probability measure is the multiplication of probability measures of individual events (see [13]).

An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B of real numbers.

In 2008, Liu [11] introduced an uncertain process, a sequence of uncertain variables indexed by time. Later on, Chen [1] investigated some properties of uncertain stationary independent increments. In 2009, Liu [12] developed an uncertain calculus based on the Liu process, a type of uncertain process that plays as an uncertain counterpart of the Brownian motion.

An uncertain process C_t is said to be a Liu process if:

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous;

(ii) C_t is an stationary independent increment process; and

(iii) every increment $C_{s+t} - C_s$ is normal uncertain variable with expected value 0 and variance t^2, whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \ x \in \mathbb{R}.$$

The Liu process holds significant importance in describing dynamic, uncertain phenomena. In order to handle the integration and differentiation of uncertain processes, Liu [12] introduced an uncertain integral with respect to the Liu process called the Liu integral.

Let $f : [0, T] \times \Gamma \to \mathbb{R}$ be an uncertain process and $C : [0, T] \times \Gamma \to \mathbb{R}$ be a Liu process. For any partition of closed interval $[0, T]$ with $0 = t_1 < t_2 < \cdots < t_n < t_{n+1} = T$, the mesh is written as $\Delta = \max\limits_{1 \le i \le n} |t_{i+1} - t_i|$. Then the Liu integral of f with respect to C is defined as

$$(\mathcal{L}iu) \int_0^T f_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{n} f_{t_i}(C_{t_{i+1}} - C_{t_i}) \tag{1}$$

provided that the limit exists almost surely (a.s.) and is finite. In this case, the uncertain process f is said to be Liu integrable.

As indicated, this definition of the Liu integral employs uniform meshes in the Riemann sums. One can consider the problem even only if the integrand has varying behavior or sharp changes in certain regions, and hence, a more precise approximation of the integral is challenging. For this reason, in this paper, we use a Henstock approach to define the new integral of an uncertain process (the integrand) with respect to ar Liu process (the integrator) called the Liu-Henstock integral. We show that the Liu-Henstock integral satisfies the standard properties of an integral.

2 Liu-Henstock Integral

In this section, we define the Liu-Henstock integral of an uncertain process with respect to a Liu-process. Throughout this paper, $\mathbb{R}_{>0}$ stands for the set of positive real numbers and $\mathbb{N}$ stands for the set of positive integers.

Definition 2.1. [16, 18] Let δ be a positive function on $[0, T)$. A finite collection $D = \{([\tau_i, v_i], \tau_i)\}_{i=1}^{n}$ of interval-point pairs is said to be *δ-fine belated partial division* of $[0, T]$ if $\{[\tau_i, v_i]\}_{i=1}^{n}$ is a collection of non-overlapping subintervals of $[0, T]$ and each $[\tau_i, v_i]$ is δ-fine belated, that is, $[\tau_i, v_i] \subset [\tau_i, \tau_i + \delta(\tau_i))$.

The term *partial* is used in Definition 2.1 since the finite collection of non-overlapping intervals $[\tau, v]$ of D may not cover the entire interval $[0, T]$. Using the Vitali covering lemma, the following concept can be defined.

Definition 2.2. [16, 18] Given a number $\eta > 0$, a given δ-fine belated partial division $D = \{([\tau, v], \tau)\}$ of $[0, T]$ is said to be *(δ, η)-fine belated partial division* of $[0, T]$ if it fails to cover $[0, T]$ by at most length η, that is,

$$\left| T - (D) \sum (v - \tau) \right| \le \eta.$$

This type of partial division is the basis to which we define the Liu-Henstock integral. Since f_t and C_t are uncertain variables for all $t \in [0, T]$, the limit in (1) is also an uncertain variable provided that the limit exists almost surely and is finite. Hence an uncertain process f is Liu integrable with respect to C if and only if the limit in (1) is an uncertain variable (see [12]). In [13], the convergence a.s. of the sequence $\{\xi_i\}$ of uncertain variables to ξ is defined in a way that there exists an event Λ with $\mathcal{M}(\Lambda) = 1$ such that $\lim\limits_{i \to \infty} |\xi_i(\gamma) - \xi(\gamma)| = 0$ for every $\gamma \in \Lambda$. In view of these, the Liu-Henstock integral is defined as follows.

Definition 2.3. Let $f : [0, T] \times \Gamma \to \mathbb{R}$ be an uncertain process. Then f is said to be *Liu-Henstock integrable* or *$\mathcal{LH}$-integrable* on $[0, T]$ with respect to C if there exists an uncertain variable L such that for every $\epsilon > 0$, there is a positive function δ on $[0, T)$ and a number $\eta > 0$ such that for any (δ, η)-fine belated partial division $D = \{([\tau, v], \tau)\}$ of $[0, T]$, we have

$$|S(f, D, \delta, \eta) - L| < \epsilon \text{ on } \Lambda$$

for some event Λ with $\mathcal{M}(\Lambda)=1$, where

$$S(f,D,\delta,\eta):=(D)\sum f_\tau(C_v-C_\tau):=(D)\sum_{i=1}^{n} f_{\tau_i}(C_{v_i}-C_{\tau_i}).$$

In this case, f is $\mathcal{LH}$-integrable to L on $[0,T]$ and L is called the $\mathcal{LH}$-integral of f which will be denoted by

$$(\mathcal{LH})\int_0^T f_t dC_t \text{ or } (\mathcal{LH})\int_0^T f dC.$$

Example 2.4. Let $C:[0,T]\times\Gamma\to\mathbb{R}$ be a Liu process. Then C is $\mathcal{LH}$-integrable on $[0,T]$ and

$$(\mathcal{LH})\int_0^T C_t dC_t=\frac{1}{2}C_T^2.$$

Let $\epsilon>0$ be given. Since almost all sample paths of C are Lipschitz continuous on $[0,T]$, there exists $k\in\mathbb{R}_{>0}$ such that

$$|C_t-C_s|\le k(t-s) \text{ on } \Lambda$$

for all $s,t\in[0,T]$ with $s<t$, and for some event Λ with $\mathcal{M}(\Lambda)=1$. Moreover, it implies that C is continuous on $[0,T]$ at each element of Λ. Since $[0,T]$ is a closed interval, by boundedness theorem, there exists $c\in\mathbb{R}_{>0}$ such that $|C_t|\le c$ on Λ for each $t\in[0,T]$. Choose $\delta(\tau)=\dfrac{\epsilon}{2k^2T}$ and $\eta=\dfrac{\epsilon}{4ck}$. Let $D=\{([\tau,v],\tau)\}$ be a (δ,η)-fine belated partial division of $[0,T]$. Let $D^c=\overline{[0,T]\setminus\bigcup_D[\tau,v]}$, i.e., the closure of $[0,T]\setminus\bigcup_D[\tau,v]$, which is the collection of closed subintervals of $[0,T]$ not in D. Then $D\cup D^c=[0,T]$ is a collection of non-overlapping subintervals of $[0,T]$. Since D is a (δ,η)-fine belated partial division of $[0,T]$, it follows that

$$(D^c)\sum(v-\tau)\le\eta.$$

Now,

$$\begin{aligned}
&\left|(D)\sum C_\tau(C_v-C_\tau)-\frac{1}{2}C_T^2\right|\\
&=\left|(D)\sum C_\tau(C_v-C_\tau)+(D)\sum-\frac{1}{2}(C_v^2-C_\tau^2)-(D)\sum-\frac{1}{2}(C_v^2-C_\tau^2)\right.\\
&\quad\left.-(D\cup D^c)\sum\frac{1}{2}(C_v^2-C_\tau^2)\right|\\
&=\left|(D)\sum\left\{C_\tau(C_v-C_\tau)-\frac{1}{2}(C_v^2-C_\tau^2)\right\}+(D)\sum\frac{1}{2}(C_v^2-C_\tau^2)\right.\\
&\quad\left.-(D)\sum\frac{1}{2}(C_v^2-C_\tau^2)-(D^c)\sum\frac{1}{2}(C_v^2-C_\tau^2)\right|\\
&=\left|(D)\sum\left\{C_\tau C_v-C_\tau^2-\frac{1}{2}C_v^2+\frac{1}{2}C_\tau^2\right\}-\frac{1}{2}(D^c)\sum(C_v^2-C_\tau^2)\right|\\
&=\left|(D)\sum\left\{-\frac{1}{2}C_v^2+C_\tau C_v-\frac{1}{2}C_\tau^2\right\}-\frac{1}{2}(D^c)\sum(C_v+C_\tau)(C_v-C_\tau)\right|\\
&\le\left|-\frac{1}{2}(D)\sum(C_v-C_\tau)^2\right|+\left|\frac{1}{2}(D^c)\sum(C_v+C_\tau)(C_v-C_\tau)\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}(D)\sum |C_v - C_\tau|^2 + \frac{1}{2}(D^c)\sum |C_v + C_\tau||C_v - C_\tau| \\
&\le \frac{k^2}{2}(D)\sum (v-\tau)(v-\tau) + ck(D^c)\sum (v-\tau) \\
&\le \frac{k^2}{2}\cdot T\delta(\tau) + ck\cdot\eta \\
&= \frac{k^2}{2}\cdot T\frac{\epsilon}{2k^2T} + ck\cdot\frac{\epsilon}{4ck} \\
&= \frac{\epsilon}{4} + \frac{\epsilon}{4} \\
&< \epsilon \text{ on } \Lambda.
\end{aligned}
$$

Hence,

$$
\left|(D)\sum C_\tau(C_v - C_\tau) - \frac{1}{2}C_T^2\right| < \epsilon \text{ on } \Lambda
$$

for some event Λ with $\mathcal{M}(\Lambda) = 1$. Therefore, C is $\mathcal{LH}$-integrable on $[0,T]$ and

$$
(\mathcal{LH})\int_0^T C_t dC_t = \frac{1}{2}C_T^2.
$$

The next example is the evaluation of Liu integral of C to the uncertain variable as in Example 2.4.

Example 2.5. [14] For any partition of $[0,T]$ with $0 = t_1 < t_2 < \cdots < t_n < t_{n+1} = T$, it follows that from (1), we have

$$
C_T^2 = \sum_{i=1}^{n}(C_{t_{i+1}}^2 - C_{t_i}^2) = \sum_{i=1}^{n}(C_{t_{i+1}} - C_{t_i})^2 + 2\sum_{i=1}^{n} C_{t_i}(C_{t_{i+1}} - C_{t_i}) \longrightarrow 0 + 2\cdot(\mathcal{L}iu)\int_0^T C_t dC_t
$$

as the mesh $\Delta \to 0$. Therefore,

$$
(\mathcal{L}iu)\int_0^T C_t dC_t = \frac{1}{2}C_T^2.
$$

3 Standard Properties

It is important to note that the Liu-Henstock integral satisfies the following standard properties of an integral namely, uniqueness of an integral, linearity, the Cauchy criterion, sequential definition, integrability on every subinterval, and the weak version of Saks-Henstock Lemma.

Proposition 3.1. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $\Lambda_1, \Lambda_2 \in \mathcal{L}$. If $\mathcal{M}(\Lambda_1) = \mathcal{M}(\Lambda_2) = 1$, then $\mathcal{M}(\Lambda_1 \cap \Lambda_2) = 1$. Consequently, $\Lambda_1 \cap \Lambda_2 \neq \varnothing$.

Proof. Suppose that $\mathcal{M}(\Lambda_1) = \mathcal{M}(\Lambda_2) = 1$. Since $\Lambda_1, \Lambda_2 \in \mathcal{L}$, it follows that $\Lambda_1 \cap \Lambda_2 \in \mathcal{L}$ and so, $\mathcal{M}(\Lambda_1 \cap \Lambda_2) \le 1$. Now,

$$
\begin{aligned}
\mathcal{M}(\Lambda_1 \cap \Lambda_2) &= 1 - \mathcal{M}((\Lambda_1 \cap \Lambda_2)^c) \\
&= 1 - \mathcal{M}(\Lambda_1^c \cup \Lambda_2^c) \\
&\ge 1 - [\mathcal{M}(\Lambda_1^c) + \mathcal{M}(\Lambda_2^c)] \\
&= 1 - [(1 - \mathcal{M}(\Lambda_1)) + (1 - \mathcal{M}(\Lambda_2))] \\
&= 1 - [(1-1) + (1-1)] = 1.
\end{aligned}
$$

Thus, $1 \leq \mathcal{M}(\Lambda_1 \cap \Lambda_2) \leq 1$ implying that $\mathcal{M}(\Lambda_1 \cap \Lambda_2) = 1$.
Suppose that $\Lambda_1 \cap \Lambda_2 = \varnothing$. Then

$$0 = \mathcal{M}(\varnothing) = \mathcal{M}(\Lambda_1 \cap \Lambda_2) = 1$$

which is a contradiction. Hence, $\Lambda_1 \cap \Lambda_2 \neq \varnothing$. □

Corollary 3.2. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $\{\Lambda_i\}_{i=1}^n \subset \mathcal{L}$. If $\mathcal{M}(\Lambda_i) = 1$ for all $i = 1, \ldots, n$, then $\mathcal{M}\left(\bigcap_{i=1}^n \Lambda_i\right) = 1$. Consequently, $\bigcap_{i=1}^n \Lambda_i \neq \varnothing$.

Proposition 3.1 and Corollary 3.2 are intended to prove the succeeding results.

Theorem 3.3. (Uniqueness) The Liu-Henstock integral is uniquely determined, in the sense that if L_1 and L_2 are two Liu-Henstock integrals of f in Definition 2.3 , then $L_1 = L_2$.

Proof. Suppose that L_1 and L_2 are two Liu-Henstock integrals of f and let $\epsilon > 0$. Then there exists a positive function δ_1 on $[0, T)$ and a number $\eta_1 > 0$ such that for any (δ_1, η_1)-fine belated partial division D_1 of $[0, T]$, we have

$$|S(f, D_1, \delta_1, \eta_1) - L_1| < \frac{\epsilon}{2} \text{ on } \Lambda_1$$

for some event Λ_1 with $\mathcal{M}(\Lambda_1) = 1$. Similarly, there exists a positive function δ_2 on $[0, T)$ and a number $\eta_2 > 0$ such that for any (δ_2, η_2)-fine belated partial division D_2 of $[0, T]$, we have

$$|S(f, D_2, \delta_2, \eta_2) - L_2| < \frac{\epsilon}{2} \text{ on } \Lambda_2$$

for some event Λ_2 with $\mathcal{M}(\Lambda_2) = 1$. Choose $\delta(\tau) = \min\{\delta_1(\tau), \delta_2(\tau)\}$ for all $\tau \in [0, T)$ and $\eta = \min\{\eta_1, \eta_2\}$. Then any (δ, η)-fine belated partial division D is also a (δ_1, η_1)-fine belated partial division and a (δ_2, η_2)-fine belated partial division of $[0, T]$. Take $\Lambda = \Lambda_1 \cap \Lambda_2$. Since Λ_1 and Λ_2 are events, Λ is also an event with $\Lambda \subset \Lambda_1$ and $\Lambda \subset \Lambda_2$. Moreover, by Proposition 3.1, $\mathcal{M}(\Lambda) = 1$ and $\Lambda \neq \varnothing$. Thus, we have

$$\begin{aligned}
|L_1 - L_2| &= |[S(f, D, \delta, \eta) - L_2] - [S(f, D, \delta, \eta) - L_1]| \\
&\leq |S(f, D, \delta, \eta) - L_2| + |S(f, D, \delta, \eta) - L_1| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon \text{ on } \Lambda.
\end{aligned}$$

Since ϵ is arbitrary, we have $|L_1(\gamma) - L_2(\gamma)| = 0$ for all $\gamma \in \Lambda$. That is, $L_1(\gamma) = L_2(\gamma)$ for all $\gamma \in \Lambda$. Since $\mathcal{M}(\Lambda) = 1$, $L_1 = L_2$. □

Theorem 3.4. (Linearity) Let $\alpha \in \mathbb{R}$. If f and g are $\mathcal{LH}$-integrable on $[0, T]$, then

(i) $f + g$ is $\mathcal{LH}$-integrable on $[0, T]$ and

$$(\mathcal{LH}) \int_0^T (f + g) dC = (\mathcal{LH}) \int_0^T f dC + (\mathcal{LH}) \int_0^T g dC;$$

(ii) αf is $\mathcal{LH}$-integrable on $[0, T]$ and

$$(\mathcal{LH}) \int_0^T \alpha f dC = \alpha \cdot (\mathcal{LH}) \int_0^T f dC.$$

Proof. Let f, g be $\mathcal{LH}$-integable on $[0, T]$ with

$$L = (\mathcal{LH}) \int_0^T f dC \text{ and } M = (\mathcal{LH}) \int_0^T g dC$$

for some uncertain variables L and M.

(i) Let $\epsilon > 0$. Then there exists a positive function δ_1 on $[0, T)$ and a number $\eta_1 > 0$ such that for any (δ_1, η_1)-fine belated partial division D_1 of $[0, T]$, we have

$$|S(f, D_1, \delta_1, \eta_1) - L| < \frac{\epsilon}{2} \text{ on } \Lambda_1$$

for some event Λ_1 with $\mathcal{M}(\Lambda_1) = 1$. Similarly, there exists a positive function δ_2 on $[0, T)$ and a number $\eta_2 > 0$ such that for any (δ_2, η_2)-fine belated partial division D_2 of $[0, T]$, we have

$$|S(g, D_2, \delta_2, \eta_2) - M| < \frac{\epsilon}{2} \text{ on } \Lambda_2$$

for some event Λ_2 with $\mathcal{M}(\Lambda_2) = 1$. Choose $\delta(\tau) = \min\{\delta_1(\tau), \delta_2(\tau)\}$ for all $\tau \in [0, T)$ and $\eta = \min\{\eta_1, \eta_2\}$. Then any (δ, η)-fine belated partial division $D = \{([\tau, v], \tau)\}$ of $[0, T]$ is a (δ_1, η_1)-fine belated partial division and a (δ_2, η_2)-fine belated partial division of $[0, T]$. Take $\Lambda = \Lambda_1 \cap \Lambda_2$. Then Λ is an event with $\Lambda \subset \Lambda_1$ and $\Lambda \subset \Lambda_2$. Moreover, by Proposition 3.1, $\mathcal{M}(\Lambda) = 1$ and $\Lambda \neq \varnothing$. Observe that

$$\begin{aligned} S(f + g, D, \delta, \eta) &= (D) \sum (f_\tau + g_\tau)(C_v - C_\tau) \\ &= (D) \sum f_\tau (C_v - C_\tau) + (D) \sum g_\tau (C_v - C_\tau) \\ &= S(f, D, \delta, \eta) + S(g, D, \delta, \eta) \text{ on } \Lambda. \end{aligned}$$

Since L and M are uncertain variables, so is $L + M$. Thus, we have

$$\begin{aligned} |S(f + g, D, \delta, \eta) - (L + M)| &= |S(f, D, \delta, \eta) + S(g, D, \delta, \eta) - (L + M)| \\ &= |S(f, D, \delta, \eta) - L + S(g, D, \delta, \eta) - M| \\ &\leq |S(f, D, \delta, \eta) - L| + |S(g, D, \delta, \eta) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ on } \Lambda. \end{aligned}$$

Since ϵ is arbitrary, $f + g$ is $\mathcal{LH}$-integrable on $[0, T]$ and

$$(\mathcal{LH}) \int_0^T (f + g) dC = L + M = (\mathcal{LH}) \int_0^T f dC + (\mathcal{LH}) \int_0^T g dC.$$

This proves the first part.

(ii) Let $\alpha \in \mathbb{R}$ and $\epsilon > 0$. Since f is $\mathcal{LH}$-integrable on $[0, T]$ with integral L, there exists a positive function δ on $[0, T)$ and a number $\eta > 0$ such that for any (δ, η)-fine belated partial division D of $[0, T]$, we have

$$|S(f, D, \delta, \eta) - L| < \frac{\epsilon}{1 + |\alpha|} \text{ on } \Lambda$$

for some event Λ with $\mathcal{M}(\Lambda) = 1$. Take the same positive function δ on $[0, T)$ and a number $\eta > 0$. Then for all (δ, η)-fine belated partial division $D = \{([\tau, v], \tau)\}$ of $[0, T]$, we have

$$S(\alpha f, D, \delta, \eta) = (D) \sum \alpha f_\tau (C_v - C_\tau)$$

$$= \alpha \cdot (D) \sum f_\tau(C_v - C_\tau)$$
$$= \alpha \cdot S(f, D, \delta, \eta) \text{ on } \Lambda.$$

Since L is an uncertain variable, $\alpha \cdot L$ is an uncertain variable for all $\alpha \in \mathbb{R}$. Thus, we have

$$\begin{aligned} |S(\alpha f, D, \delta, \eta) - \alpha \cdot L| &= |\alpha \cdot S(f, D, \delta, \eta) - \alpha \cdot L| \\ &= |\alpha \cdot [S(f, D, \delta, \eta) - L]| \\ &= |\alpha| \cdot |S(f, D, \delta, \eta) - L| \\ &< \frac{|\alpha| \cdot \epsilon}{1 + |\alpha|} < \epsilon \text{ on } \Lambda. \end{aligned}$$

Since ϵ is arbitrary, αf $\mathcal{LH}$-integrable on $[0, T]$ and

$$(\mathcal{LH}) \int_0^T \alpha f dC = \alpha \cdot L = \alpha \cdot (\mathcal{LH}) \int_0^T f dC.$$

This proves the theorem. □

Theorem 3.5. [20] Let $\{\xi_i\}$ be a sequence of uncertain variables and $\lim_{i\to\infty} \xi_i = \xi$ almost surely. Then ξ is an uncertain variable.

Definition 3.6. Let $\{\xi_i\}$ be an uncertain sequence. Then we call the $\{\xi_i\}$ a *Cauchy sequence a.s.* if for every $\epsilon > 0$, there exists an event Λ with $\mathcal{M}(\Lambda) = 1$ and $N \in \mathbb{N}$ such that for every $\gamma \in \Lambda$, we have

$$|\xi_i(\gamma) - \xi_j(\gamma)| < \epsilon$$

for any $i, j \geq N$.

Lemma 3.7. Let $\{\xi_i\}$ be a sequence of uncertain variables. If $\{\xi_i\}$ is a Cauchy sequence a.s., then $\{\xi_i\}$ converges to some uncertain variable ξ a.s..

Proof. Let $\{\xi_i\}$ be a Cauchy sequence a.s. and $\epsilon > 0$. Then there exists an event Λ with $\mathcal{M}(\Lambda) = 1$ and $N \in \mathbb{N}$ such that for all $\gamma \in \Lambda$, we have

$$|\xi_i(\gamma) - \xi_j(\gamma)| < \frac{\epsilon}{2}$$

for all $i, j \geq N$. This implies that for any fixed $\gamma = \gamma_0 \in \Lambda$,

$$|\xi_i(\gamma_0) - \xi_j(\gamma_0)| < \frac{\epsilon}{2} < \epsilon$$

for all $i, j \geq N$. Since $\{\xi_i\}$ is a sequence of real-valued functions, $\{\xi_i(\gamma_0)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $k \in \mathbb{R}$ such that $\xi_i(\gamma_0) \longrightarrow k$ as $i \longrightarrow \infty$. In this way, define $\xi : \Gamma \to \mathbb{R}$ by $\gamma \mapsto \lim_{j\to\infty} \xi_j(\gamma)$. Since $\Lambda \subset \Gamma$, it follows that for all $\epsilon_0 > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $\gamma \in \Lambda$, we have

$$|\xi_j(\gamma) - \xi(\gamma)| < \epsilon_0$$

for all $j \geq N_0$. Now, for all $\gamma \in \Lambda$,

$$\begin{aligned} |\xi_i(\gamma) - \xi(\gamma)| &= |\xi_i(\gamma) - \xi_j(\gamma) + \xi_j(\gamma) - \xi(\gamma)| \\ &\leq |\xi_i(\gamma) - \xi_j(\gamma)| + |\xi_j(\gamma) - \xi(\gamma)| \end{aligned}$$

$$< \frac{\epsilon}{2} + \epsilon_0$$

which implies that

$$|\xi_i(\gamma) - \xi(\gamma)| \leq \frac{\epsilon}{2} < \epsilon$$

for all $i \geq N$. Since $\mathcal{M}(\Lambda) = 1$, $\xi_i \longrightarrow \xi$ a.s., and by Theorem 3.5, ξ is an uncertain variable. This completes the proof. □

Theorem 3.8. (Cauchy Criterion) An uncertain process $f : [0, T] \times \Gamma \to \mathbb{R}$ is $\mathcal{LH}$-integrable on $[0, T]$ if and only if for every $\epsilon > 0$, there exists a positive function δ on $[0, T)$ and a positive number η such that for any two (δ, η)-fine belated partial divisions D and D' of $[0, T]$, we have

$$\left|S(f, D, \delta, \eta) - S(f, D', \delta, \eta)\right| < \epsilon \text{ on } \Lambda$$

for some event Λ with $\mathcal{M}(\Lambda) = 1$.

Proof. Suppose that f is $\mathcal{LH}$-integrable on $[0, T]$ with integral L. Let $\epsilon > 0$. Then there exists a positive function δ on $[0, T)$ and a number $\eta > 0$ such that for any (δ, η)-fine belated partial divisions D and D' of $[0, T]$, we have

$$|S(f, D, \delta, \eta) - L| < \frac{\epsilon}{2} \text{ on } \Lambda \text{ and } \left|S(f, D', \delta, \eta) - L\right| < \frac{\epsilon}{2} \text{ on } \Lambda$$

for some event Λ with $\mathcal{M}(\Lambda) = 1$. Thus, we have

$$\begin{aligned} \left|S(f, D, \delta, \eta) - S(f, D', \delta, \eta)\right| &= \left|S(f, D, \delta, \eta) - L + L - S(f, D', \delta, \eta)\right| \\ &= \left|S(f, D, \delta, \eta) - L - (S(f, D', \delta, \eta) - L)\right| \\ &\leq |S(f, D, \delta, \eta) - L| + \left|S(f, D', \delta, \eta) - L\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ on } \Lambda. \end{aligned}$$

Hence, $|S(f, D, \delta, \eta) - S(f, D', \delta, \eta)| < \epsilon$ on Λ for some event Λ with $\mathcal{M}(\Lambda) = 1$.

To prove the converse, note that for each $n \in \mathbb{N}$, there exists a positive function δ_n on $[0, T)$ and a positive number η_n such that for all (δ_n, η_n)-fine belated partial divisions D and D' of $[0, T]$, we have

$$\left|S(f, D, \delta_n, \eta_n) - S(f, D', \delta_n, \eta_n)\right| < \frac{1}{n} \text{ on } \Lambda_1$$

for some event Λ_1 with $\mathcal{M}(\Lambda_1) = 1$. We may assume that for all $n \in \mathbb{N}$, $\delta_n(x) \geq \delta_{n+1}(x)$ for all $x \in [0, T)$; otherwise, we replace δ_n by $\tilde{\delta}_n = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$ on $[0, T)$, for all $n \in \mathbb{N}$ and also choose $\{\eta_n\}$ to be decreasing. For each $n \in \mathbb{N}$, let D_n be a (δ_n, η_n)-fine belated partial division of $[0, T]$. Since $\delta_n(x) \geq \delta_{n+1}(x)$ for all $x \in [0, T)$, for any $n, m \in \mathbb{N}$ with $m \geq n$, D_m and D_n are both (δ_n, η_n)-fine belated partial division of $[0, T]$. Hence,

$$|S(f, D_n, \delta_n, \eta_n) - S(f, D_m, \delta_m, \eta_m)| < \frac{1}{n} \text{ on } \Lambda_1.$$

Thus, given any $\epsilon > 0$, by taking $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$, we have

$$|S(f, D_n, \delta_n, \eta_n) - S(f, D_m, \delta_m, \eta_m)| < \frac{1}{n} \leq \frac{1}{N} < \epsilon \text{ on } \Lambda_1$$

for all $m, n \geq N$. This means that $\{S(f, D_n, \delta_n, \eta_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence a.s. of uncertain variables. By Lemma 3.7, there exists an uncertain variable L and an event Λ_2 with $\mathcal{M}(\Lambda_2) = 1$ such that

$$\lim_{n\to\infty} S(f, D_n, \delta_n, \eta_n) = L \text{ on } \Lambda_2.$$

Let $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$

$$|S(f, D_n, \delta_n, \eta_n) - L| < \frac{\epsilon}{2} \text{ on } \Lambda_2.$$

Furthermore, there exists $N_2 \in \mathbb{N}$ such that $N_2 > \frac{2}{\epsilon}$. Take $N = \max\{N_1, N_2\}$. Choose $\delta = \delta_N$ on $[0, T)$ and $\eta = \eta_N$. Then any (δ, η)-fine belated partial division D of $[0, T]$ is also a (δ_N, η_N)-fine belated partial division of $[0, T]$. Take $\Lambda = \Lambda_1 \cap \Lambda_2$. Then Λ is an event with $\Lambda \subset \Lambda_1$ and $\Lambda \subset \Lambda_2$. Moreover, by Proposition 3.1, $\mathcal{M}(\Lambda) = 1$ and $\Lambda \neq \varnothing$. Thus, we have

$$\begin{aligned}
|S(f, D, \delta, \eta) - L| &= |S(f, D, \delta, \eta) - S(f, D_N, \delta_N, \eta_N) + S(f, D_N, \delta_N, \eta_N) - L| \\
&\leq |S(f, D, \delta, \eta) - S(f, D_N, \delta_N, \eta_N)| + |S(f, D_N, \delta_N, \eta_N) - L| \\
&< \frac{1}{N} + \frac{\epsilon}{2} \\
&\leq \frac{1}{N_2} + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ on } \Lambda
\end{aligned}$$

Hence, f is $\mathcal{LH}$-integrable to L on $[0, T]$. □

Theorem 3.9. A function $f : [0, T] \times \Gamma \to \mathbb{R}$ is $\mathcal{LH}$-integrable on $[0, c]$ and $[c, T]$ where $c \in (0, T)$ if and only if f is $\mathcal{LH}$-integrable on $[0, T]$ and

$$(\mathcal{LH}) \int_0^T f \, dC = (\mathcal{LH}) \int_0^c f \, dC + (\mathcal{LH}) \int_c^T f \, dC.$$

Proof. Suppose that $f : [0, T] \times \Gamma \to \mathbb{R}$ is $\mathcal{LH}$-integrable on $[0, c]$ and $[c, T]$ where $c \in (0, T)$ with

$$L = (\mathcal{LH}) \int_0^c f dC \text{ and } M = (\mathcal{LH}) \int_c^T f dC.$$

Let $\epsilon > 0$. Then there exists a positive function δ_1 on $[0, c)$ and a number $\eta_1 > 0$ such that for any (δ_1, η_1)-fine belated partial division D_1 of $[0, c]$, we have

$$|S(f, D_1, \delta_1, \eta_1) - L| < \frac{\epsilon}{2} \text{ on } \Lambda_1$$

for some event Λ_1 with $\mathcal{M}(\Lambda_1) = 1$. Similarly, there exists a positive function δ_2 on $[c, T)$ and a number $\eta_2 > 0$ such that for any (δ_2, η_2)-fine belated partial division D_2 of $[c, T]$, we have

$$|S(f, D_2, \delta_2, \eta_2) - M| < \frac{\epsilon}{2} \text{ on } \Lambda_2$$

for some event Λ_2 with $\mathcal{M}(\Lambda_2) = 1$. Define δ on $[0, T)$ by

$$\delta(x) = \begin{cases} \min\{\delta_1(x), c - x\} & \text{if } x \in [0, c) \\ \delta_2(x) & \text{if } x \in [c, T) \end{cases}$$

and take $\eta = \min\{\eta_1, \eta_2\}$. Let $D = \{([\tau, v], \tau)\}$ be any (δ, η)-fine belated partial division of $[0, T]$. Consider

$$P_1 = \{([\tau, v], \tau) \in D : [\tau, v] \subset [0, c]\} \text{ and } P_2 = \{([\tau, v], \tau) \in D : [\tau, v] \subset [c, T]\}$$

Observe that

$$\begin{aligned}
\left|T - (D)\sum(v-\tau)\right| &= \left|c + (T-c) - ((P_1 \cup P_2)\sum(v-\tau))\right| \\
&= \left|c + (T-c) - ((P_1)\sum(v-\tau) + (P_2)\sum(v-\tau))\right| \\
&= \left|c + (T-c) - (P_1)\sum(v-\tau) - (P_2)\sum(v-\tau)\right| \\
&= \left|c - (P_1)\sum(v-\tau) + (T-c) - (P_2)\sum(v-\tau)\right| \\
&= \left|c - (P_1)\sum(v-\tau)\right| + \left|(T-c) - (P_2)\sum(v-\tau)\right|
\end{aligned}$$

Then P_1 is a δ_1-fine belated partial division of $[0, c]$ and

$$\left|c - (P_1)\sum(v-\tau)\right| \leq \left|T - (D)\sum(v-\tau)\right| \leq \eta \leq \eta_1$$

so that P_1 is a (δ_1, η_1)-fine belated partial division of $[0, c]$. Similarly, P_2 is a δ_2-fine belated partial division of $[c, T]$ and

$$\left|(T-c) - (P_2)\sum(v-\tau)\right| \leq \left|T - (D)\sum(v-\tau)\right| \leq \eta \leq \eta_2$$

so that P_2 is a (δ_2, η_2)-fine belated partial division of $[c, T]$. Take $\Lambda = \Lambda_1 \cap \Lambda_2$. Then Λ is an event with $\Lambda \subset \Lambda_1$ and $\Lambda \subset \Lambda_2$. Moreover, by Proposition 3.1, $\mathcal{M}(\Lambda) = 1$ and $\Lambda \neq \varnothing$. Note that

$$\begin{aligned}
S(f, D, \delta, \eta) &= (D)\sum f_\tau(C_v - C_\tau) \\
&= (P_1 \cup P_2)\sum f_\tau(C_v - C_\tau) \\
&= (P_1)\sum f_\tau(C_v - C_\tau) + (P_2)\sum f_\tau(C_v - C_\tau) \\
&= S(f, P_1, \delta_1, \eta_1) + S(f, P_2, \delta_2, \eta_2) \text{ on } \Lambda.
\end{aligned}$$

Since L and M are uncertain variables, so is $L + M$. Hence, we have

$$\begin{aligned}
|S(f, D, \delta, \eta) - (L+M)| &= |S(f, P_1, \delta_1, \eta_1) + S(f, P_2, \delta_2, \eta_2) - (L+M)| \\
&= |S(f, P_1, \delta_1, \eta_1) - L + S(f, P_2, \delta_2, \eta_2) - M| \\
&= |S(f, P_1, \delta_1, \eta_1) - L| + |S(f, P_2, \delta_2, \eta_2) - M| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ on } \Lambda.
\end{aligned}$$

Since ϵ is arbitrary, f is $\mathcal{LH}$-integrable on $[0, T]$ and

$$(\mathcal{LH})\int_0^T f dC = L + M = (\mathcal{LH})\int_0^c f dC + (\mathcal{LH})\int_c^T f dC.$$

To prove the converse, we shall show that f is $\mathcal{LH}$-integrable on $[0, c]$. Let f be $\mathcal{LH}$-integrable on $[0, T]$ and let $\epsilon > 0$ be given. By Cauchy Criterion, there exists a positive function δ on

$[0,T)$ and a positive number η such that for any (δ,η)-fine belated partial divisions D and D' of $[0,T]$, we have

$$\big|S(f,D,\delta,\eta)-S(f,D',\delta,\eta)\big|<\epsilon \text{ on } \Lambda$$

for some event Λ with $\mathcal{M}(\Lambda)=1$. Choose $\eta'=\frac{\eta}{2}$. Then any (δ,η')-fine belated partial division $D_1=\{([\tau_i,v_i],\tau_i)\}_{i=1}^{n}$ of $[0,T]$ is a (δ,η)-fine belated partial division of $[0,T]$. Hence, we may assume that $([\tau_g,c],\tau_g)\in D_1$ for some $g\in\{1,2,\dots,n\}$. Let D_1' be the collection of interval-point pairs in D_1 by deleting $([\tau_g,c],\tau_g)$ and $\{([\tau_i,v_i],\tau_i)\}_{i=1}^{g-1}$ from D_1. This means that D_1' is a (δ,η')-fine belated partial division of $[c,T]$. Let D_2 and D_2' be any (δ,η')-fine belated partial divisions of $[0,c]$. Then $D_2\cup D_1'$ and $D_2'\cup D_1'$ are δ-fine belated partial divisions of $[0,T]$. Moreover,

$$\begin{aligned}\Big|T-(D_2\cup D_1')\sum(v-\tau)\Big| &= \Big|c+(T-c)-(D_2)\sum(v-\tau)-(D_1')\sum(v-\tau)\Big|\\ &= \Big|c-(D_2)\sum(v-\tau)\Big|+\Big|(T-c)-(D_1')\sum(v-\tau)\Big|\\ &\le \eta'+\eta'=\eta\end{aligned}$$

so that $D_2\cup D_1'$ is a (δ,η)-fine belated partial division of $[0,T]$. Similarly, $D_2'\cup D_1'$ is also a (δ,η)-fine belated partial division of $[0,T]$. Then

$$\begin{aligned}&\big|S(f,D_2,\delta,\eta')-S(f,D_2',\delta,\eta')\big|\\ &=\Big|\Big(S(f,D_2,\delta,\eta')+(D_1')\sum f_\tau(C_v-C_\tau)\Big)-\Big((D_1')\sum f_\tau(C_v-C_\tau)+S(f,D_2',\delta,\eta')\Big)\Big|\\ &=\Big|\Big((D_2)\sum f_\tau(C_v-C_\tau)+(D_1')\sum f_\tau(C_v-C_\tau)\Big)-\Big((D_1')\sum f_\tau(C_v-C_\tau)\\ &\qquad +(D_2')\sum f_\tau(C_v-C_\tau)\Big)\Big|\\ &=\Big|(D_2\cup D_1')\sum f_\tau(C_v-C_\tau)-(D_1'\cup D_2')\sum f_\tau(C_v-C_\tau)\Big|\\ &=\big|S(f,D_2\cup D_1',\delta,\eta)-S(f,D_2'\cup D_1',\delta,\eta)\big|<\epsilon \text{ on } \Lambda\end{aligned}$$

for some event Λ with $\mathcal{M}(\Lambda)=1$. Applying the Cauchy Criterion, f is $\mathcal{LH}$-integrable on $[0,c]$. Following the same argument above, we can also verify that f is $\mathcal{LH}$-integrable on $[c,T]$. ☐

Theorem 3.10. (Sequential Definition) An uncertain process $f:[0,T]\times\Gamma\to\mathbb{R}$ is $\mathcal{LH}$-integrable on $[0,T]$ if and only if there exists an uncertain variable L, a decreasing sequence of $\{\delta_n\}$ of positive functions defined on $[0,T)$ and a decreasing sequence of positive numbers $\{\eta_n\}$ such that for any (δ_n,η_n)-fine belated partial division D_n of $[0,T]$, we have

$$\lim_{n\to\infty}|S(f,D_n,\delta_n,\eta_n)-L|=0 \text{ on } \Lambda$$

for some event Λ with $\mathcal{M}(\Lambda)=1$. In this case,

$$L=(\mathcal{LH})\int_0^T f_t dC_t.$$

Proof. Suppose that f is $\mathcal{LH}$-integrable on $[0,T]$ with integral L. Then for each $n\in\mathbb{N}$, there exists a positive function δ_n on $[0,T)$ and a positive number η_n such that for any (δ_n,η_n)-fine belated partial division D_n of $[0,T]$,

$$|S(f,D_n,\delta_n,\eta_n)-L|<\frac{1}{n} \text{ on } \Lambda$$

for some event Λ with $\mathcal{M}(\Lambda) = 1$. We can choose $\{\delta_n(\tau)\}$ and $\{\eta_n\}$ to be decreasing for all $\tau \in [0, T)$. Hence,

$$\lim_{n\to\infty} |S(f, D_n, \delta_n, \eta_n) - L| = 0 \text{ on } \Lambda$$

for some event Λ with $\mathcal{M}(\Lambda) = 1$.

Conversely, assume that there exists an uncertain variable L, a decreasing sequence $\{\delta_n(\tau)\}$ of positive functions defined on $[0, T)$ and a decreasing sequence of positive numbers $\{\eta_n\}$ such that for any (δ_n, η_n)-fine belated partial division D_n of $[0, T]$, we have

$$\lim_{n\to\infty} |S(f, D_n, \delta_n, \eta_n) - L| = 0 \text{ on } \Lambda_0$$

for some event Λ_0 with $\mathcal{M}(\Lambda_0) = 1$. Suppose that f is not $\mathcal{LH}$-integrable to L on $[0, T]$. Then there exists $\epsilon > 0$ such that for every positive function δ on $[0, T)$ and every number $\eta > 0$, there exists (δ, η)-fine belated partial division D of $[0, T]$ with

$$\left|(D)\sum f_\tau(\gamma)(C_v(\gamma) - C_\tau(\gamma)) - L(\gamma)\right| \geq \epsilon$$

for some $\gamma \in \Lambda$, for all events Λ with $\mathcal{M}(\Lambda) = 1$. This means that for each positive function δ_n on $[0, T)$ and number $\eta_n > 0$, there exists a (δ_n, η_n)-fine belated partial division D_n of $[0, T]$ with

$$\left|(D_n)\sum f_\tau(\gamma)(C_v(\gamma) - C_\tau(\gamma)) - L(\gamma)\right| \geq \epsilon$$

for some $\gamma \in \Lambda$, for all events Λ with $\mathcal{M}(\Lambda) = 1$. Hence,

$$\lim_{n\to\infty} \left|(D_n)\sum f_\tau(\gamma)(C_v(\gamma) - C_\tau(\gamma)) - L(\gamma)\right| \neq 0$$

for some event $\gamma \in \Lambda$, for all events Λ with $\mathcal{M}(\Lambda) = 1$. This gives a contradiction. □

Theorem 3.11. If $f : [0, T] \times \Gamma \to \mathbb{R}$ is $\mathcal{LH}$-integrable on $[0, T]$, then f is also $\mathcal{LH}$-integrable on every subinterval $[c, d]$ of $[0, T]$.

Proof. Supoose $f : [0, T] \times \Gamma \to \mathbb{R}$ is $\mathcal{LH}$-integrable on $[0, T]$ and let $c, d \in (0, T)$ with $c < d$. By Theorem 3.9, f is $\mathcal{LH}$-integrable on $[0, c]$ and $[0, d]$. Let

$$(\mathcal{LH})\int_0^c f dC = L \text{ and } (\mathcal{LH})\int_0^d f dC = M$$

for some uncertain variables L and M. Let $\epsilon > 0$ be given. Then there exists positive function δ_1 on $[0, T)$ and a number $\eta_1 > 0$ such that for any (δ_1, η_1)-fine belated partial division D_1 of $[0, c]$, we have

$$|S(f, D_1, \delta_1, \eta_1) - L| < \frac{\epsilon}{2} \text{ on } \Lambda_1$$

for some event Λ_1 with $\mathcal{M}(\Lambda_1) = 1$. Similarly, there exists a positive function δ_2 on $[0, T)$ and a number $\eta_2 > 0$ such that for any (δ_2, η_2)-fine belated partial division D_2 of $[0, d]$, we have

$$|S(f, D_2, \delta_2, \eta_2) - M| < \frac{\epsilon}{2} \text{ on } \Lambda_2$$

for some event Λ_2 with $\mathcal{M}(\Lambda_2) = 1$. Choose $\delta = \min\{\delta_1, \delta_2\}$ on $[0, T)$ and $\eta = \min\{\eta_1, \frac{\eta_2}{2}\}$. Let D be a (δ, η)-fine belated partial division of $[c, d]$ and let D_1' be a (δ, η)-fine belated partial division of $[0,c]$. Then $D_2' = D_1' \cup D$ is a $(\delta, 2\eta)$-fine belated partial division of $[0,d]$ and thus a (δ_2, η_2)-fine belated partial division of $[0,d]$. Note that $D = D_2' \setminus D_1'$. Take $\Lambda = \Lambda_1 \cap \Lambda_2$. Then Λ is an event with $\Lambda \subset \Lambda_1$ and $\Lambda \subset \Lambda_2$. Moreover, by Proposition 3.1, $\mathcal{M}(\Lambda) = 1$ and $\Lambda \neq \varnothing$. Since L and M are uncertain variables, so is $M - L$. Now,

$$\begin{aligned}
|S(f,D,\delta,\eta) - (M-L)| &= \left|S(f, D_2' \setminus D_1', \delta, \eta) - (M-L)\right| \\
&= \left|S(f, D_2', \delta_2, \eta_2) - S(f, D_1', \delta_1, \eta_1) - (M-L)\right| \\
&= \left|S(f, D_2', \delta_2, \eta_2) - M - (S(f, D_1', \delta_1, \eta_1) - L)\right| \\
&\leq \left|S(f, D_2', \delta_2, \eta_2) - M\right| + \left|S(f, D_1', \delta_1, \eta_1) - L\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ on } \Lambda.
\end{aligned}$$

Therefore, f is $\mathcal{LH}$-integrable on $[c,d] \subset [0,T]$ and

$$(\mathcal{LH}) \int_c^d f dC = M - L = (\mathcal{LH}) \int_0^d f dC - (\mathcal{LH}) \int_0^c f dC$$

which completes the proof. □

Theorem 3.12. (Saks-Henstock Lemma (Weak Version)) Let f be $\mathcal{LH}$-integrable on $[0,T]$ and $F[u,v] := (\mathcal{LH}) \int_u^v f_t dC_t$ for any $[u,v] \subseteq [0,T]$. Then for every $\epsilon > 0$, there exists a positive function δ on $[0,T)$ and a positive number η such that for any (δ,η)-fine belated partial division $D = \{([\tau,v],\tau)\}$ of $[0,T]$, we have

$$\left|(D) \sum \{f_\tau (C_v - C_\tau) - F[\tau, v]\}\right| < \epsilon \text{ on } \Lambda$$

for some event Λ with $\mathcal{M}(\Lambda) = 1$.

Proof. Let $\epsilon > 0$ be given. Then there exists a positive function δ on $[0,T)$ and a positive number η such that for any (δ,η)-fine belated partial division P of $[0,T]$, we have

$$\left|S(f,P,\delta,\eta) - (\mathcal{LH}) \int_0^T f_t dC_t\right| < \frac{\epsilon}{2} \text{ on } \Lambda_0$$

for some event Λ_0 with $\mathcal{M}(\Lambda_0) = 1$. Let $D = \{([\tau_i, v_i], \tau_i)\}_{i=1}^n$ be a (δ,η)-fine belated partial division of $[0,T]$. Consider $[0,T] \setminus \bigcup_{i=1}^n [\tau_i, v_i]$, which consists of disjoint intervals of the form (a_j, b_j), $j = 1, \ldots, m$. Note that f is also $\mathcal{LH}$-integrable on $[a_j, b_j]$ for all j. This means that for all j, there exists a positive function δ_j on $[a_j, b_j)$ and a positive number η_j such that for any (δ_j, η_j)-fine belated partial division D_j of $[a_j, b_j]$, we have

$$|S(f, D_j, \delta_j, \eta_j) - F[a_j, b_j]| < \frac{\epsilon}{2^{j+1}} \text{ on } \Lambda_j$$

for some event Λ_j with $\mathcal{M}(\Lambda_j) = 1$. We choose $\{\delta_j\}_{j=1}^m$ and $\{\eta_j\}_{j=1}^m$ such that $\delta_j(\tau) \leq \delta(\tau)$ for all j and $\sum_{j=1}^m \eta_j \leq \eta$. Let $P = D \cup D_1 \cup D_2 \cup \ldots D_m$ which is a (δ,η)-fine belated partial division of $[0,T]$. Observe that

$$\left|S(f,P,\delta,\eta) - (\mathcal{LH}) \int_0^T f_t dC_t\right|$$

$$= \left|(P)\sum f_\tau(C_v - C_\tau) - \left[\sum_{i=1}^{n}(\mathcal{LH})\int_{\tau_i}^{v_i} f_t dC_t + \sum_{j=1}^{m}(\mathcal{LH})\int_{a_j}^{b_j} f_t dC_t\right]\right|$$

$$= \left|(P)\sum f_\tau(C_v - C_\tau) - \sum_{i=1}^{n} F[\tau_i, v_i] - \sum_{j=1}^{m} F[a_j, b_j]\right|$$

$$= \left|(D \cup D_1 \cup D_2 \cup \ldots D_m)\sum f_\tau(C_v - C_\tau) - \sum_{i=1}^{n} F[\tau_i, v_i] - \sum_{j=1}^{m} F[a_j, b_j]\right|$$

$$= \left|(D)\sum f_\tau(C_v - C_\tau) + \left(\bigcup_{j=1}^{m} D_j\right)\sum f_\tau(C_v - C_\tau) - \sum_{i=1}^{n} F[\tau_i, v_i] - \sum_{j=1}^{m} F[a_j, b_j]\right|$$

$$= \left|(D)\sum\{f_\tau(C_v - C_\tau) - F[\tau, v]\} + \sum_{j=1}^{m}\{S(f, D_j, \delta_j, \eta_j) - F[a_j, b_j]\}\right|.$$

It follows that

$$\left|(D)\sum\{f_\tau(C_v - C_\tau) - F[\tau, v]\} + \sum_{j=1}^{m}\{S(f, D_j, \delta_j, \eta_j) - F[a_j, b_j]\}\right| < \frac{\epsilon}{2} \text{ on } \Lambda_0$$

for some event Λ_0 with $\mathcal{M}(\Lambda_0) = 1$. Take $\Lambda = \Lambda_0 \cap \left(\bigcap_{j=1}^{m} \Lambda_j\right)$. Then Λ is an event with $\Lambda \subset \Lambda_0$ and $\Lambda \subset \left(\bigcap_{j=1}^{m} \Lambda_j\right)$. Moreover, by Corollary 3.2, $\mathcal{M}(\Lambda) = 1$ and $\Lambda \neq \varnothing$. Thus, we have

$$\begin{aligned}
&\left|(D)\sum\{f_\tau(C_v - C_\tau) - F[\tau, v]\}\right| \\
&= \left|(D)\sum\{f_\tau(C_v - C_\tau) - F[\tau, v]\} + \sum_{j=1}^{m}\{S(f, D_j, \delta_j, \eta_j) - F[a_j, b_j]\}\right. \\
&\qquad \left. - \sum_{j=1}^{m}\{S(f, D_j, \delta_j, \eta_j) - F[a_j, b_j]\}\right| \\
&\leq \left|(D)\sum\{f_\tau(C_v - C_\tau) - F[\tau, v]\} + \sum_{j=1}^{m}\{S(f, D_j, \delta_j, \eta_j) - F[a_j, b_j]\}\right| \\
&\qquad + \left|\sum_{j=1}^{m}\{S(f, D_j, \delta_j, \eta_j) - F[a_j, b_j]\}\right| \\
&\leq \left|(D)\sum\{f_\tau(C_v - C_\tau) - F[\tau, v]\} + \sum_{j=1}^{m}\{S(f, D_j, \delta_j, \eta_j) - F[a_j, b_j]\}\right| \\
&\qquad + \sum_{j=1}^{m}|S(f, D_j, \delta_j, \eta_j) - F[a_j, b_j]| \\
&< \frac{\epsilon}{2} + \sum_{j=1}^{m}\frac{\epsilon}{2^{j+1}}
\end{aligned}$$

$$
\begin{aligned}
&< \frac{\epsilon}{2} + \sum_{j=1}^{\infty} \frac{\epsilon}{2^{j+1}} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \sum_{j=1}^{\infty} \frac{1}{2^{j}} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ on } \Lambda.
\end{aligned}
$$

This completes the proof. □

Conclusion and Recommendation

In this paper, we use a Henstock approach to define a new integral for the uncetain process with respect to a Liu process. This newly defined integral, called Liu-Henstock integral adheres to the standard properties of an integral. A worthwhile direction for further investigation is to formulate a version of Itô's formula for the Liu-Henstock integral.

Acknowledgement

The authors are grateful to the anonymous referees for their helpful comments and suggestions.

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