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Integrable Set and Measurable Function

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Abstract: In this paper, using concept of integrable sets we prove that a McShane integrable function is measurable.

Keywords/phrases: McShane integral, characteristic function, integrable set, Lebesgue outer measure, measurable function.

1 Introduction

The classical measure theory was developed form the concept of Lebesgue outer measure. In $\mathbb{R}$, Lebesgue outer measure allows us to extend the notion of length of an interval and measure theory provides the useful abstraction of the notion of length of subsets of the real line and, more generally, area and volume of subsets of Euclidean space $\mathbb{R}^n$. In particular, it provided a systematic answer to the question of which subsets of $\mathbb{R}$ have length.

In [9], Yang investigated measure theory independently of Lebesgue using the well-developed theory of Henstock integration. By utilizing the same view, in this note, we study measurable functions.

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2 Preliminary Results

Definition 2.1 A function $f : [a, b] \to \mathbb{R}$ is said to be *McShane* (*resp., Henstock*) *integrable to a real number A on* $[a, b]$ if for any $\epsilon > 0$, there exists a function $\delta : [a, b] \to \mathbb{R}^+$ such that for any McShane (resp., Henstock) δ-fine division $D = \{([u, v], \xi)\}$ of $[a, b]$, we have

$$\left|(D) \sum f(\xi)(v - u) - A\right| < \epsilon.$$

If $f : [a, b] \to \mathbb{R}$ is McShane (resp., Henstock) integrable to A on $[a, b]$, then we write

$$A = (\mathcal{M}) \int_a^b f \quad \Big(\text{resp., } A = (\mathcal{H}) \int_a^b f\Big).$$

By a McShane (resp., Henstock) δ-fine division $D = \{([u, v]; \xi)\}$ of $[a, b]$ we mean that $[u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ (resp., $\xi \in [u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$), for all $([u, v]; \xi) \in D$.

Theorem 2.2 [3] If $f : [a, b] \to \mathbb{R}$ is McShane integrable on $[a, b]$, then f is Henstock integrable on $[a, b]$.

Theorem 2.3 [3] If $f : [a, b] \to \mathbb{R}$ is Henstock integrable and non-negative on $[a, b]$, then f is McShane integrable on $[a, b]$.

Hence, McShane and Henstock integrals are equivalent if the function under consideration is non-negative.

Theorem 2.4 (Monotone Convergence Theorem)[3] *Let $\{f_n\}_{n=1}^{\infty}$ be an increasing sequence of McShane integrable functions on $[a, b]$ and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for each $x \in [a,b]$. *If* $\lim_{n\to\infty} (\mathcal{M}) \int_a^b f_n = A$, *then* f *is McShane integrable on* $[a,b]$ *and*

$$(\mathcal{M}) \int_a^b f = A = \lim_{n\to\infty} (\mathcal{M}) \int_a^b f_n.$$

If $\{f_n\}_{n=1}^{\infty}$ is decreasing, then by considering the sequence $\{-f_n\}_{n=1}^{\infty}$, an analogous result also holds for decreasing sequence. Theorem 2.5 also holds for Henstock integral.

Definition 2.5 Let $E \subseteq \mathbb{R}$ and the function $\mathbf{1}_E : \mathbb{R} \to \mathbb{R}$ define by

$$\mathbf{1}_E = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

is called the *characteristic function* on E.

Definition 2.6 Let $E \subseteq \mathbb{R}$. The *Lebesgue outer measure* of E, $m^*(E)$, is given by

$$m^*(E) = \inf \left\{ \sum_{k=1}^{+\infty} \ell(I_k) : \ A \subseteq \bigcup_{k=1}^{+\infty} I_k \right\},$$

where $\{I_k : k \in \mathbb{N}\}$ is a countable collection of non-empty open and bounded intervals I_k.

Definition 2.7 A set $E \subseteq \mathbb{R}$ is *Lebesgue measurable*, or simply *measurable*, if for any set $A \subseteq \mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c),$$

where m^* is the Lebesgue outer measure.

We denote by $\mathcal{M}$ the collection of all measurable subsets $E \subseteq \mathbb{R}$. If $E \in \mathcal{M}$, we define

$$m(E) = m^*(E).$$

Definition 2.8 A set $E \subseteq \mathbb{R}$ is *integrable* if $\mathbf{1}_{E\cap[a,b]}$ is McShane integrable on $[a,b]$, for all $[a,b] \subset \mathbb{R}$.

Remark 2.9 The following statements are immediate from the definition:

(i) The sets $\varnothing$ and $\mathbb{R}$ are integrable sets.

(ii) If E is integrable, then the complement E^c of E is also integrable.

(iii) The collection of all integrable sets forms a σ-algebra on $\mathbb{R}$.

Theorem 2.10 [9] *Let $X \subseteq [a,b]$. The following statements are equivalent:*

(i) *The characteristic function $\mathbf{1}_X$ on X is McShane integrable on $[a,b]$.*

(ii) *X is Lebesgue measurable.*

Corollary 2.11 *Let X be a bounded subset of $\mathbb{R}$. If X is an integrable set, then X is Lebesgue measurable.*

Proof: Let $a < b$ be real numbers such that $X \subseteq [a,b]$ and X is an integrable set. Then $\mathbf{1}_{X\cap[c,d]}$ is McShane integrable on $[c,d]$ for all $[c,d] \subseteq \mathbb{R}$. In particular, $\mathbf{1}_{X\cap[a,b]} = \mathbf{1}_X$ is McShane integrable on $[a,b]$. Thus, by Theorem 2.10, X is Lebesgue measurable. $\square$

Theorem 2.12 [6] *Let the function f have a measurable domain E. The the following statements are equivalent:*

(i) *For each real number c, the set $\{x \in E : f(x) > c\}$ is measurable.*

(*ii*) *For each real number* c*, the set* $\{x \in E : f(x) \geq c\}$ *is measurable.*

(*iii*) *For each real number* c*, the set* $\{x \in E : f(x) < c\}$ *is measurable.*

(*iv*) *For each real number* c*, the set* $\{x \in E : f(x) \leq c\}$ *is measurable.*

Each of these properties implies that for each extended real number c*, the set* $\{x \in E : f(x) = c\}$ *is measurable.*

Definition 2.13 [6] An extended real-valued function f defined on E is said to be *Lebesgue measurable*, or simply *measurable*, provided its domain E is measurable and it satisfies one of the four statements of Theorem 2.12.

3 Results

First, we define *variation zero* and *almost everywhere.*

Definition 3.1 An integrable set $E \subseteq [a, b]$ is said to have variation zero if

$$(\mathcal{M}) \int_a^b \mathbf{1}_E(x) = 0.$$

It is worth noting that a subset of a set of variation zero is again of variation zero.

Definition 3.2 A property is said to hold everywhere (abbreviated *a.e.*) on A if the set of points in A where it fails to hold is an integrable set of variation zero.

The proof of the following result can be found in [1]. It says that every Mcshane integrable functions on $[a, b]$ can be approximated by step functions.

Lemma 3.3 [1] *If $f : [a,b] \to \mathbb{R}$ is McShane integrable on $[a,b]$, then there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of step functions such that $\varphi_n \to f$ a.e. on $[a,b]$ and*

$$\lim_{n\to\infty} (\mathcal{M}) \int_a^b |\varphi_n - f| = 0.$$

Theorem 3.4 *Let c be any real number. If $f : [a,b] \to \mathbb{R}$ is McShane integrable on $[a,b]$ and $X = \{x \in [a,b] : f(x) < c\}$, then the characteristic function $\mathbf{1}_X$ on X is McShane integrable.*

Proof: By Lemma 3.3, there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of step functions such that $\varphi_n \to f$ on $[a,b]$ except on a set S of variation zero. Let

$$X(\varphi_i < c - \tfrac{1}{k}) = \{x \in [a,b] : \varphi_i(x) < c - \tfrac{1}{k}\}$$

where $c \in \mathbb{R}$. Then for each i and k, the function $\mathbf{1}_{X(\varphi_i < c - \frac{1}{k})}$ is McShane integrable on $[a,b]$, being a step function. Thus, $\mathbf{1}_{\bigcap_{i=n}^{m} X(\varphi_i < c - \frac{1}{k})}$ is also McShane integrable on $[a,b]$, for any m. Let

$$Y = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} X(\varphi_i < c - \tfrac{1}{k}).$$

Note that if $A \subseteq B$, then $\mathbf{1}_A \leq \mathbf{1}_B$. Hence, for any fixed n

$$\mathbf{1}_{\bigcap_{i=n}^{m+1} X(\varphi_i < c - \frac{1}{k})} \leq \mathbf{1}_{\bigcap_{i=n}^{m} X(\varphi_i < c - \frac{1}{k})},$$

for each $m \geq n$ since

$$\bigcap_{i=n}^{m+1} X(\varphi_i < c - \tfrac{1}{k}) \subseteq \bigcap_{i=n}^{m} X(\varphi_i < c - \tfrac{1}{k}).$$

It can be seen that for each $x \in [a,b]$,

$$\lim_{m\to\infty} \mathbf{1}_{\bigcap_{i=n}^{m} X(\varphi_i < c - \frac{1}{k})}(x) = \mathbf{1}_{\bigcap_{i=n}^{\infty} X(\varphi_i < c - \frac{1}{k})}(x).$$

Hence, $\{\mathbf{1}_{\bigcap_{i=n}^{m} X(\varphi_i<c-\frac{1}{k})}\}_m$ is a decreasing sequence of McShane integrable functions on $[a,b]$. Thus, by the Monotone Convergence Theorem, $\mathbf{1}_{\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}$ is McShane integrable on $[a,b]$, for each n.

On other hand, $\{\mathbf{1}_{\bigcup_{n=1}^{m}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}\}_{m=1}^{\infty}$ is an increasing sequence of McShane integrable functions on $[a,b]$ and for each $x \in [a,b]$, we have

$$\lim_{m\to\infty} \mathbf{1}_{\bigcup_{n=1}^{m}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}(x) = \mathbf{1}_{\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}(x).$$

Again, by the Monotone Convergence Theorem,

$$\mathbf{1}_{\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}$$

is McShane integrable on $[a,b]$.

Similarly, $\{\mathbf{1}_{\bigcup_{k=1}^{m}\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}\}_{m=1}^{\infty}$ is an increasing sequence of McShane integrable functions on $[a,b]$ and for each $x \in [a,b]$

$$\begin{aligned}\lim_{m\to\infty} &\mathbf{1}_{\bigcup_{k=1}^{m}\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}(x) \\ &= \mathbf{1}_{\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}(x) = \mathbf{1}_Y\end{aligned}$$

and hence, by the Monotone Convergence Theorem, $\mathbf{1}_Y$ is McShane integrable on $[a,b]$.

It can be seen that $X \setminus S = Y \setminus S$, implying that $X \setminus Y \subseteq S$. Hence, $X \setminus Y$ is also a set of variation zero. Therefore, the characteristic function $\mathbf{1}_X$ on X is McShane integrable on $[a,b]$. $\square$

Similarly, if $f : [a,b] \to \mathbb{R}$ is McShane integrable on $[a,b]$, then the characteristic function on $\{x \in [a,b] : f(x) > c\}$ is also McShane integrable on $[a,b]$. By Remark 2.9, with the same f, the characteristic functions on

$$\{x \in [a,b] : f(x) \leq c\} \text{ and}$$

$$\{x \in [a, b] : f(x) \geq c\}$$

are also McShane integrable on $[a, b]$.

Corollary 3.5 *If $f : [a, b] \to \mathbb{R}$ is McShane integrable on $[a, b]$, then the set*

$$X = \{x \in [a, b] : f(x) < c\}$$

is Lebesgue measurable for any $c \in \mathbb{R}$.

Proof: By Theorem 3.4, the characteristic function $\mathbf{1}_X$ is McShane integrable on $[a, b]$. Hence, by Theorem 2.10, X is Lebesgue measurable. □

Corollary 3.6 *If $f : [a, b] \to \mathbb{R}$ is McShane integrable on $[a, b]$, then f is measurable.*

Proof: By Corollary 3.5, the set $\{x \in [a, b] : f(x) < c\}$ is an integrable set for all $c \in \mathbb{R}$. Therefore by Definition 2.13, f is measurable. □

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