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On the AB-Generalized Fibonacci, Pell and Jacobsthal Sequences by Hessenberg Matrices

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Abstract: In this paper, we introduce the second order linear recurrence relation of the AB-generalized Fibonacci sequence $\{u_n\}$ and give the relationships between $\{u_n\}$ and Hessenberg permanents and determinants. Moreover, we also give representations of u_{2n} and u_{2n+1}. These formulas generalize the one obtained earlier by Kiliç et al. in [9].

Keywords/phrases: Fibonacci sequence, Pell sequence, Jacobsthal sequence, generalized Fibonacci sequence, AB- generalized Fibonacci sequence.

1 Introduction

For $n > 0$, the Fibonacci sequence $\{F_n\}$ is defined by

$$F_{n+1} = F_n + F_{n-1},$$

where $F_0 = 0$ and $F_1 = 1$, the Pell sequence $\{P_n\}$ is defined by

$$P_{n+1} = 2P_n + P_{n-1},$$

where $P_0 = 0$ and $P_1 = 1$, and the Jacobsthal sequence is defined by

$$J_{n+1} = J_n + 2J_{n-1},$$

where $J_0 = 0$ and $J_1 = 1$.

In [5], Kiliç introduced the generalized Fibonacci sequence and gave the explicit formulas for the sums of the terms of this sequence using matrix methods. He constructed essential generating matrices and used matrix properties to obtain these sums. Kiliç's definition provided a motivation to the construction of the so called AB-generalized Fibonacci sequence.

Let $n > 0$ and let A and B be nonzero integers with $A^2 + 4B \neq 0$. The AB**-generalized Fibonacci sequence** $\{u_n\}$ has the recurrence relation

$$u_{n+1} = Au_n + Bu_{n-1},$$

where $u_0 = 0$ and $u_1 = 1$.

Let α and β be the roots of the characteristic equation $x^2 - Ax - B = 0$. Then the Binet's formula of the sequence $\{u_n\}$ has the form

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

In [1], the author gave the combinatorial representation of $\{u_n\}$ and is given by

$$u_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} A^{n-2k} B^k.$$

Several authors have studied the second order linear recurrences and show their relationships between the permanents and determinants of tridiagonal matrices.

In [10], the authors gave interesting results involving the permanent of the $(-1, 0, 1)$-matrix and the Fibonacci number F_{n+1}. Consequently, the authors established some results involving the positively and negatively subscripted terms of the Fibonacci and Lucas numbers.

In [8], the authors discovered the families of $(0,1)$-matrices and then gave the relationships between the permanents of these matrices and the sums of the Fibonacci and Lucas numbers.

In [4], the author introduced two tridiagonal matrices and then gave the relationships between the permanents and determinants of these matrices and the second order linear recurrences.

In [11], the authors introduced the two generalized doubly stochastic matrices and then show the relationships between the doubly stochastic permanents and the second order linear recurrences.

Recently, the authors in [9], define lower Hessenberg matrices and gave the relationships between the permanents and determinants of these matrices and the generalized Fibonacci and Pell sequences.

A lower Hessenberg matrix $M_n = (a_{ij})$ is an $n \times n$ matrix where $a_{jk} = 0$ whenever $k > j+1$ and $a_{j(j+1)} \neq 0$ for some j. Clearly,

$$M_n = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & \vdots & 0 \\ a_{31} & a_{32} & a_{33} & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{(n-2)(n-1)} & 0 \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(n-2)} & a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n(n-1)} & a_{nn} \end{bmatrix}.$$

In [2], the authors considered the above general lower Hessenberg matrix and then gave the following determinant formula: For $n \geq 2$,

$$\det M_n = a_{nn} \cdot \det M_{n-1} + \sum_{r=1}^{n-1} \left((-1)^{n-r} a_{mr} \prod_{j=r}^{n-1} a_{j(j+1)} \det M_{r-1} \right).$$

In this paper, we consider the AB-generalized Fibonacci sequence $\{u_n\}$ and then we show the relationships between the AB-generalized Fibonacci sequence and the Hessenberg determinants and permanents. Also, we give the representations of u_{2n} and u_{2n+1}. The authors in [9] proved the results for special case $B = 1$.

2 On the AB-Generalized Fibonacci sequence by Hessenberg matrices

In this section, we consider first the following Hessenberg matrices. Let the $n \times n$ lower Hessenberg matrix H_n defined by

$$H_n = \begin{bmatrix} A^2+B & B & 0 & \cdots & 0 & 0 \\ B & A^2+B & B & \ddots & \vdots & 0 \\ B & B & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & B & 0 \\ B & B & \cdots & B & A^2+B & B \\ B & B & B & \cdots & B & A^2+B \end{bmatrix}.$$

We also define another $n \times n$ lower Hessenberg matrix T_n by

$$T_n = \begin{bmatrix} A^2+B & B & 0 & \cdots & 0 & 0 \\ B & A^2+B & B & \ddots & \vdots & 0 \\ B & B & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & B & 0 \\ B & B & \cdots & B & A^2+B & B \\ B & B & B & \cdots & B & B \end{bmatrix}.$$

Following the methods employed by the authors in [9], we have the following results.

Lemma 2.1 *For every $n \geq 3$,*

$$\det T_n = A^2 B \det H_{n-2}.$$

Proof: We subtract first the $(n-1)^{th}$ row from the n^{th} row and then expanding with respect to the last row, we can easily obtain $\det T_n = A^2 B \det H_{n-2}$. □

Theorem 2.2 *For every $n > 0$,*

$$u_{n+2} = \frac{\det H_n}{A^{n-1}} \quad or \quad \det H_n = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} A^{2n-2k} B^k.$$

Proof: The equation holds for $n = 1$. Now, assume that $u_{n+2} = \dfrac{\det H_n}{A^{n-1}}$. We now show that the equation holds for $n+1$. By the cofactor expansion along the last column, we have $\det H_{n+1} = (A^2 + B) \det H_n - B \det T_n$. By Lemma 2.2 and using the assumption, we have

$$\det H_{n+1} = (A^2 + B) \det H_n - A^2 B^2 \det H_{n-2} = A^n u_{n+3}.$$

Thus, by induction, the assertion must be true. □

If $A = B = 1$, then the sequence $\{u_n\}$ is a Fibonacci sequence $\{F_n\}$, and using Theorem 2.2, we have

$$\begin{vmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \ddots & \vdots & 0 \\ 1 & 1 & 2 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{vmatrix}_{n \times n} = F_{n+2},$$

which can be found in [2].

Moreover, if $A = 1$ and $B = 2$, then the sequence $\{u_n\}$ is a Jacobsthal sequence $\{J_n\}$. Similarly, using Theorem 2.2, we have

$$\begin{vmatrix} 3 & 2 & 0 & \cdots & 0 & 0 \\ 2 & 3 & 2 & \ddots & \vdots & 0 \\ 2 & 2 & 3 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 & 0 \\ 2 & 2 & \cdots & 2 & 3 & 2 \\ 2 & 2 & 2 & \cdots & 2 & 3 \end{vmatrix}_{n\times n} = J_{n+2}.$$

As done by the authors in [9], we shall now consider the permanent of a Hessenberg matrix. We define first the following concepts.

A matrix M is said to be *convertible* if there is an $n \times n$ $(1, -1)$-matrix H such that per $M = \det(M \circ H)$, where $M \circ H$ denotes the Hadamard product of M and H. The matrix H is called the *converter* of M.

Now, let S be an $n \times n$ $(1, -1)$-matrix defined by

$$S = \begin{bmatrix} 1 & -1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & -1 & \ddots & \vdots & 1 \\ 1 & 1 & 1 & \ddots & 1 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

We denote the Hadamard product $H_n \circ S$ by C_n. Then

$$C_n = \begin{bmatrix} A^2+B & -B & 0 & \cdots & 0 & 0 \\ B & A^2+B & -B & \ddots & \vdots & 0 \\ B & B & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -B & 0 \\ B & B & \cdots & B & A^2+B & -B \\ B & B & B & \cdots & B & A^2+B \end{bmatrix}.$$

Then, we have the following result which is a consequence of Theorem 2.2.

Corollary 2.3 *For every $n > 0$,*

$$u_{n+2} = \frac{\text{per } C_n}{A^{n-1}} \text{ or per } C_n = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} A^{2n-2k} B^k.$$

If $A = 2$ and $B = 1$, then the sequence $\{u_n\}$ is a Pell sequence $\{P_n\}$, and using Corollary 2.4, we have

$$\text{per} \begin{bmatrix} 5 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 5 & -1 & \ddots & \vdots & 0 \\ 1 & 1 & 5 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ 1 & 1 & \cdots & 1 & 5 & -1 \\ 1 & 1 & 1 & \cdots & 1 & 5 \end{bmatrix}_{n \times n} = 2^{n-1} P_{n+2}.$$

3 Representations of u_{2n} and u_{2n+1}

In this section, we give the representations of u_{2n} and u_{2n+1} using permanents and determinants of some Hessenberg matrices.

Firstly, let the $n \times n$ lower Hessenberg matrix W_n defined by

$$W_n = \begin{bmatrix} A^2+B & -B & 0 & \cdots & 0 & 0 \\ A^2 & A^2+B & -B & \ddots & \vdots & 0 \\ A^2 & A^2 & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -B & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+B & -B \\ A^2 & A^2 & A^2 & \cdots & A^2 & A^2+B \end{bmatrix}.$$

Then, we have the following result.

Theorem 3.1 *For every $n > 0$,*

$$u_{2n+1} = \det W_n.$$

Proof: The equation holds when $n = 1$. Assume that $u_{2n+1} = \det W_n$. We now show that the equation also holds for $n+1$. Now, subtracting the n^{th} row from the $(n+1)^{th}$ row and expanding along the last column, we have $\det W_{n+1} = (A^2 + B) \det W_n - B^2 \det W_{n-1}$. By the assumption and the recurrence relation of the sequence $\{u_n\}$, we have $\det W_{n+1} = u_{2n+3}$. Thus, by induction, the assertion must be true. $\Box$

Secondly, Let the $n \times n$ lower Hessenberg matrix V_n defined by

$$V_n = \begin{bmatrix} A^2 & -B & 0 & \cdots & 0 & 0 \\ A^2 & A^2+B & -B & \ddots & \vdots & 0 \\ A^2 & A^2 & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -B & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+B & -B \\ A^2 & A^2 & A^2 & \cdots & A^2 & A^2+B \end{bmatrix}.$$

Then, we have the following result.

Theorem 3.2 *For every $n > 0$,*

$$u_{2n} = \frac{\det V_n}{A}.$$

Proof: The equation is clearly true if $n = 1$. Assume that $u_{2n} = \frac{\det V_n}{A}$. We now show that the equation also holds for $n + 1$. Then expanding along the first row, we have $\det V_{n+1} = A^2 \det W_n + B \det V_n$. By assumption, recurrence relation of the sequence $\{u_n\}$ and Theorem 3.3, we have

$$\det V_{n+1} = A^2 u_{2n+1} + ABu_{2n} = Au_{2n+2}.$$

Hence, by induction, the result follows. □

Consider again the $n \times n$ $(1, -1)$ matrix S defined previously. We denote the Hadamard products $W_n \circ S$ and $V_n \circ S$ by R_n and Q_n, respectively. Then

$$R_n = \begin{bmatrix} A^2+B & B & 0 & \cdots & 0 & 0 \\ A^2 & A^2+B & B & \ddots & \vdots & 0 \\ A^2 & A^2 & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & B & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+B & B \\ A^2 & A^2 & A^2 & \cdots & A^2 & A^2+B \end{bmatrix}$$

and

$$Q_n = \begin{bmatrix} A^2 & B & 0 & \cdots & 0 & 0 \\ A^2 & A^2+B & B & \ddots & \vdots & 0 \\ A^2 & A^2 & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & B & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+B & B \\ A^2 & A^2 & A^2 & \cdots & A^2 & A^2+B \end{bmatrix}.$$

Then we have the following results which are consequences of Theorems 3.3 and 3.4.

Corollary 3.3 *For every $n > 0$,*

$$u_{2n+1} = \text{per } R_n.$$

Corollary 3.4 *For every $n > 0$,*

$$u_{2n} = \frac{\text{per } Q_n}{A}.$$

Using the above results and identity in [2], we have the following representations:

$$\det W_n = \text{per } R_n = \sum_{k=0}^{n} \binom{2n-k}{k} A^{2n-2k} B^k,$$

and

$$\det V_n = \text{per } Q_n = \sum_{k=0}^{\lfloor \frac{2n-1}{2} \rfloor} \binom{2n-1-k}{k} A^{2n-2k} B^k.$$

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