Volume 3 Issue 1 May 2012

The MINDANAWAN
Journal of Mathematics

ISSN 2094-7380

McShane Integral in Locally Convex Topological Vector Spaces

Juniven P. Acapulco[a], **Julius V. Benitez**[b]
[a]*MSU-General Santos*, [b]*MSU-Iligan Institute of Technology*
[a]ajuniven@gmail.com;
[b]julius.benitez@g.msuiit.edu.ph

Abstract: In this note, we show that the definition of the McShane integral in [2] coincides with the definition of the McShane integral introduced in [3] whenever the functions are taking values in a locally convex topological vector space.

Keywords/phrases: topological vector space, locally convex, Mcshane integral

1 Introduction

In the late 1960's, Edward James McShane introduced an integral which was a modified version of the Henstock integral and proved that its equivalent to the Lebesgue integral. He broadened the class of tagged partitions by not insisting that the tag of an interval belong to the interval. In 1990, Gordon [1] extend the definition of the McShane integral for functions taking values in Banach spaces and since then attention has been paid to this field. Paluga [2] also introduced the definition of the McShane integral for functions taking values in topological vector spaces (TVS). He proved that the TVS version of the McShane integral coincides with the Banach version whenever the functions are taking values in Banach spaces. Just recently, Tato and Temaj [3] defined the McShane integral for functions taking values in locally convex topological vector spaces (LCTVS).

Here we prove that the McShane integral introduced in

[2] and [3] coincide whenever the functions take values in LCTVS. Minkowski functionals play a major role in establishing the objective of this paper.

2 Preliminaries

Definition 2.1 Let A be a non-empty subset of a real vector space X. Then

(i) A is **convex** if $tA + (1-t)A \subseteq A$ for any $t \in [0,1]$.

(ii) A is **balanced** if $tA \subseteq A$ for any $t \in [-1,1]$.

(iii) A is **absorbing** if for each $x \in X$ there exists $s > 0$ such that for any $t \in \mathbb{R}$ with $|t| > s$, we have $x \in tA$.

Definition 2.2 Let A be a non-empty absorbing subset of a real vector space X. The **Minkowski's functional** of A is given by the function

$$\begin{array}{rcrcl} \mu_A & : & X & \to & [0, +\infty) \\ & & x & \mapsto & \inf A_x \end{array}$$

where $A_x = \{k > 0 : x \in kA\}$.

Definition 2.3 Let X be a real vector space. A function $\rho : X \to \mathbb{R}$ is said to be a **seminorm** on X if

(i) $\rho(x+y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$,

(ii) $\rho(\alpha x) = |\alpha|\rho(x)$ for all $x \in X$ and all $\alpha \in \mathbb{R}$.

If ρ has the additional property that $\rho(x) = 0$ implies $x = \theta$, then ρ is a **norm** on X.

Definition 2.4 Let X be a real vector space. A family $\mathcal{P}$ of seminorms on X is said to be **separating** if for every $x \in X$ with $x \neq \theta$, there exists $\rho \in \mathcal{P}$ such that $\rho(x) \neq 0$.

Definition 2.5 Suppose $\mathcal{T}$ is a topology on a real vector space X such that

(i) $\{x\}$ is closed for every $x \in X$ and

(ii) the vector space operations are continuous with respect to $\mathcal{T}$.

Under these conditions, $\mathcal{T}$ is said to be a **vector topology** on X and X is a **topological vector space**.

Althroughout the paper, θ-nbd W means that W is open in a TVS X, and $\theta \in W$, where θ is the zero vector in X.

Definition 2.6 Let X be a topological vector space. We say that the function $f : X \to \mathbb{R}$ is **continuous** at $c \in X$ if for every $\epsilon > 0$ we can find a θ-nbd V such that for any x in X with $x - c \in V$, we have $|f(x) - f(c)| < \epsilon$.

Remark 2.7 If X is a TVS, then continuity of the vector space operations mean that the functions

$$\begin{array}{rcl} + : X \times X & \to & X \\ (x, y) & \mapsto & x + y \end{array} \quad \text{and} \quad \begin{array}{rcl} \cdot : \mathbb{R} \times X & \to & X \\ (\alpha, x) & \mapsto & \alpha x \end{array}$$

are continuous where $\mathbb{R}$ carries the usual topology and the spaces $\mathbb{R} \times X$ and $X \times X$ carry their respective product topologies.

Definition 2.8 Let X be a TVS. Then X is said to be **locally convex** if there exists a local base $\mathcal{B}$ at the zero vector θ whose members are convex. In this case, we say that X is **locally convex topological vector space** (LCTVS).

The following results are useful in this paper. See [4] for details.

Theorem 2.9 [4] *Let X be a TVS, $a \in X$ and $t \neq 0$. Then for any open set G in X, $a+G$ and tG are also open in X.*

Theorem 2.10 [4] *Let X be a TVS. Then every θ-nbd is absorbing.*

Theorem 2.11 [4] *If X is LCTVS, then there exists a local base $\mathcal{B}$ at θ whose members are both convex and balanced.*

Theorem 2.12 [4] *Suppose A is a convex balanced absorbing set in a real vector space X. Then μ_A is a seminorm on X.*

Theorem 2.13 [4] *Suppose $\mathcal{P}$ is a separating family of seminorms on a real vector space X. For each $\rho \in \mathcal{P}$ and each $\epsilon > 0$, let*

$$V(\rho, \epsilon) = \{x \in X : \rho(x) < \epsilon\}.$$

Let $\mathcal{B}(\mathcal{P})$ be the collection of all finite intersections of the sets $V(\rho, \epsilon)$. Define

$$\mathcal{T}(\mathcal{P}) = \{G \subseteq X : \forall x \in G, \exists B \in \mathcal{B}(\mathcal{P}) \text{ such that } x+B \subseteq G\}.$$

Then $\mathcal{T}(\mathcal{P})$ is a vector topology on X and $\mathcal{B}(\mathcal{P})$ is a convex balanced local base for $\mathcal{T}(\mathcal{P})$, which turns X into LCTVS with the property that every member of $\mathcal{P}$ is $\mathcal{T}(\mathcal{P})$-continuous.

Definition 2.14 A **division** of the interval $[a, b]$ is a finite collection $\{I_i : 1 \leq i \leq n\}$ of non-overlapping closed intervals I_i such that $[a, b] = \bigcup_{i=1}^{n} I_i$. Let δ be a positive function defined on $[a, b]$. We say that the collection $D = \{(I_i, \xi_i) : 1 \leq i \leq n\}$ of interval-point pairs is a **δ-fine free tagged division** of $[a, b]$ if

(*i*) $\{I_i : 1 \leq i \leq n\}$ is a division of $[a, b]$, (*ii*) $I_i \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in [a,b]$ for $i = 1, \cdots, n$.

We shall denote the **length** of I_i by $|I_i|$.

Definition 2.15 [2] Let X be a topological vector space. A function $f : [a,b] \to X$ is **McShane integrable** on $[a,b]$ if there is an $\alpha \in X$ such that for any θ-nbd W, there is a positive function δ on $[a,b]$ such that for any δ-fine free tagged division $D = \{(I_i, \xi_i) : 1 \le i \le n\}$ of $[a,b]$, we have

$$f(D) - \alpha \in W$$

where $f(D) = \sum_{i=1}^{n} f(\xi_i)|I_i|$. In this case, we call α the **McShane integral** of f on $[a,b]$ and we write

$$(\mathcal{M}) \int_a^b f = \alpha.$$

3 Main Result

First, we consider the following lemmas.

Lemma 3.1 *Let X be a real vector space and let $A \subseteq X$. If A is convex and absorbing then for each $r > 0$, we have*

$$\{x \in X : \mu_A(x) < r\} \subseteq rA.$$

Proof: Since A is absorbing, using Definition 2.1(*iii*), we get $\theta \in A$. Let $x \in X$ with $\mu_A(x) < r$. We will show that $x \in rA$. By Definition 2.2, $\mu_A(x) = \inf A_x$, where $A_x = \{k > 0 : x \in kA\}$. Thus, there must exists $t \in A_x$ such that $t < r$. So, $t > 0$ and $x \in tA$. Now, $0 < tr^{-1} < 1$ and since A is convex with $\theta \in A$, it follows from Definition 2.1(*i*) that $tr^{-1}A \subseteq A$. Thus, $x \in tA \subseteq rA$. Hence, $x \in rA$. The result follows. □

Lemma 3.2 *Suppose A is an open convex balanced θ-nbd in a TVS X. Then μ_A is continuous.*

Proof: Since A is a θ-nbd, by Theorem 2.10, A is absorbing. So, μ_A is well defined. Using Theorem 2.12, the Minkowski functional μ_A is a seminorm on X. Let $\epsilon > 0$. Take $V = \epsilon 2^{-1}A$. Then by Theorem 2.9, V is a θ-nbd. Thus for any x in X with $x - c \in V$, we get

$$|\mu_A(x) - \mu_A(c)| \leq \mu_A(x - c) \leq \frac{\epsilon}{2} < \epsilon.$$

Hence, by Definition 2.6, the function μ_A is continuous. $\square$

In what follows, $(X, \mathcal{T})$ is a locally convex topological vector space and $\mathcal{P}(X)$ is a family of $\mathcal{T}$-continuous seminorms on X so that the topology is generated by $\mathcal{P}(X)$.

Definition 3.3 [3] A function $f : [a, b] \to X$ is said to be **McShane integrable** on $[a, b]$ if there exists $\alpha \in X$ such that for each $\rho \in \mathcal{P}(X)$ and for any $\epsilon > 0$ there exists $\delta_\rho : [a, b] \to (0, +\infty)$ such that if $D = \{(I_i, \xi_i) : 1 \leq i \leq n\}$ is a δ_ρ-fine free tagged division of $[a, b]$, we have

$$\rho(f(D) - \alpha) < \epsilon.$$

We now state and prove our main result.

Theorem 3.4 *Let X be LCTVS. Then Definitions* 2.15 *and* 3.3 *are equivalent.*

Proof: First, we will show that Definition 2.15 implies Definition 3.3. Assume that f is McShane integrable on $[a, b]$ in the sense of Definition 2.15. Let $\epsilon > 0$ and let $\rho \in \mathcal{P}(X)$. Take

$$W = \{x \in X : \rho(x) < 1\}.$$

Since ρ is a seminorm on X, $\rho(\theta) = 0 < 1$ and so, $\theta \in W$. Now,

$$W = \rho^{-1}((-\infty, 1))$$

and because ρ is $\mathcal{T}$-continuous, $W \in \mathcal{T}$. Thus W is a θ-nbd. Thus, there exists a gauge $\delta_\rho : [a, b] \to (0, +\infty)$ such that if $D = \{(I_i, \xi_i) : 1 \leq i \leq n\}$ is a δ_ρ-fine free tagged division of $[a, b]$, we have

$$f(D) - (\mathcal{M}) \int_a^b f \in \epsilon W.$$

This implies that

$$\rho\left(f(D) - (\mathcal{M}) \int_a^b f\right) < \epsilon.$$

Next, assume that f is McShane integrable on $[a, b]$ in the sense of Definition 3.3. Let W be θ-nbd. Then by Theorem 2.11, there exists a convex balanced θ-nbd V such that $V \subseteq W$. Since V is a θ-nbd, by Theorem 2.10, V is absorbing. So, μ_V is well defined. Thus, by using Theorem 2.12, μ_V is a seminorm on X. Also, by Lemma 3.2, μ_V is $\mathcal{T}$-continuous. Hence, $\mu_V \in \mathcal{P}(X)$. Thus, corresponding to $\epsilon = 1$, there exists a gauge $\delta : [a, b] \to (0, +\infty)$ such that if $D = \{(I_i, \xi_i) : 1 \leq i \leq n\}$ is a δ-fine free tagged division of $[a, b]$, we have

$$\mu_V(f(D) - \alpha) < 1.$$

Note that by Lemma 3.1, we get

$$\{x \in X : \mu_V(x) < 1\} \subseteq V.$$

Thus, $f(D) - \alpha \in V \subseteq W$. □

References

[1] R.A. Gordon, *The McShane integral of Banach-valued functions*, Illinois J. Math, 34(1990), 557-567.

[2] S.R. Canoy and R.N. Paluga, *On the McShane integral in topological vector spaces*, Chamchuri Journal of Mathematics, Vol.2(2010)No.2, pp.45-56.

[3] A.Tato and I. Timaj, *The Convergence Theorems for the McShane Integral of Functions Taking Values in a Locally Convex Space*, Int. Journal of Math. Analysis, Vol.6(2012)No.40, pp.1965-1976.

[4] W. Rudin, *Functional Analysis*, McGraw-Hill Inc, 1973.