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The Average of the mth Power of the L_m-norms of Littlewood Polynomials on the Unit Circle

Braullo D. Peñalosa[a], **Jocelyn P. Vilela**[b]

[a]*Department of Mathematics, College of Arts and Sciences, Central Mindanao University, Musuan, Bukidnon*

[b]*Department of Mathematics, College of Science and Mathematics, MSU-Iligan Institute of Technology, Tibanga, Iligan City*

Abstract: Let $n \geq 0$ be any integer and

$$\mathfrak{L}_n = \left\{ P : P(z) = \sum_{j=0}^{n} a_j z^j, a_j \in \{1, -1\} \right\}$$

be a set of polynomials of degree n. The elements of the set $\mathfrak{L}_n$ are restricted polynomials called Littlewood polynomial. If $P \in \mathfrak{L}_n$, then the L_m-norm of P over the unit circle is

$$\|P(z)\|_m = \left(\frac{1}{2\pi} \int_0^{2\pi} |P(z)|^m \, d\theta \right)^{\frac{1}{m}}, \quad (z = e^{i\theta})$$

and the average of the mth power of the L_m-norms over $\mathfrak{L}_n$ is

$$\mu_n(m) = \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P\|_m^m .$$

The formulae for $\mu_n(m)$ for $m = 2$, 4, 6, and 8 have been established in the literature by Borwein and Choi in their paper entitled "The Average Norm of Polynomials of Fixed Height".

In this paper we give exact formulae for $\mu_n(m)$ for $m = 10$. This result is new and is the tip of an iceberg that we explore further.

1 Introduction

Questions and results about polynomials with restricted coefficients has been given much attention in recent years. Among the well-known and well-studied polynomials are the Newmann polynomials [13], the Littlewood polynomials [4, 5, 8], and the fixed height one polynomials [3, 12]. Some authors like Peter Borwein and Kwok-Kwong Stephen Choi [3] give a variety of related results for different classes of polynomials including polynomials of fixed height H, polynomials with coefficients of modulus one, derivative polynomials and reciprocal polynomials.

Problems concerning the location and multiplicity of zeros of polynomials with restricted coefficients, some of the approximation theoretic properties of such polynomials, on the maximum and minimum norms of such polynomials, and finding the average norm of such polynomials are only few questions in literature that arose in the above mention polynomials.

The results in [3] which deals with the average norm of Littlewood polynomials are already established in 2005 by Toufik Mansour [12]. His proof and techniques, without doubt are significantly more complicated and requires a considerable foundation in the area of generating functions and knowledge in using the MAPLE program explicitly. With the aid of MAPLE software, it was then that the results in [12] were implemented and produced a lot of outputs.

However, the exact formula for the average norm of Littlewood polynomials are provided with proof in an entirely different approach in [3] in an article entitled " The Average Norm of Polynomials of Fixed Height". This paper then enumerates the results in [3] which determines the average of the mth power of the L_m-norms of Littlewood polynomials for an even integer m where $2 \leq m \leq 8$. Furthermore, the author extended this study to the case $m = 10$.

2 Preliminary Concepts and Known Results

Definition 2.1 [3] Let $n \geq 0$ be any integer. A polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$, where $a_j \in \{1, -1\}$, is a restricted polynomial called Littlewood polynomial.

The results obtain about Littlewood polynomials by Mansour [12] presented below are provided with algebraic proofs by Borwein and Choi [3]. These results are so critical that we need them to enumerate in this section. We also enumerate all lemmas in [3] for these are badly needed for the proof of the main result. They are the following.

Lemma 2.2 *For $m \neq 0$, we have*

$$\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} \left|P'(z)\right|^2 z^m \; d\theta = 0.$$

Lemma 2.3 *For an integer $m \geq 1$, we have*

$$\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} \left|P'(z)\right|^2 z^m P'(z)^2 \; d\theta = 0.$$

Lemma 2.4 *For an integer $m \geq 0$, we have*

$$\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} P'(\overline{z})^2 \; d\theta = \begin{cases} 1 & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

Lemma 2.5 *For an integer $m \geq 1$, we have*

$$\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} \left|P'(z)\right|^4 \, d\theta = \begin{cases} n - m + 1 & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

Lemma 2.6 *For an integer $m \geq 1$, we have*

$$\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} \left|P'(z)\right|^4 z^{2m} P'(z)^2 \, d\theta$$

$$= \begin{cases} \frac{3}{2}(n-m)^2 + \frac{7}{2}(n-m) + \frac{3-(-1)^{m+n}}{2} & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

The following result are called the average of the second, fourth, sixth, and eighth power of the norms of Littlewood polynomials on the unit circle, respectively.

Theorem 2.7 *For $n \geq 0$, we have*

$$\begin{aligned} \mu_n(2) &= n + 1, \\ \mu_n(4) &= 2n^2 + 3n + 1, \\ \mu_n(6) &= 6n^3 + 9n^2 + 4n + 1, \text{ and} \\ \mu_n(8) &= 24n^4 + 30n^3 + 4n^2 + 5n + 4 - 3(-1)^n. \end{aligned}$$

3 Main Results

The formula for $\mu_n(10)$ is somewhat more complicated. We proceed as follows.

Lemma 3.1 *For $m \in \mathbb{Z}$, we have*

$$\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} \left|P'(z)\right|^2 z^m \, d\theta = \begin{cases} 0 & \text{if } m \neq 0, \\ n + 1 & \text{if } m = 0. \end{cases}$$

Proof: The case when $m \neq 0$ is the same as Lemma 2.2. Suppose $m = 0$. We prove this case by induction on n. When $n = 0$, we have

$$\frac{1}{2^{0+1}} \sum_{P' \in \mathfrak{L}_0} \frac{1}{2\pi} \int_0^{2\pi} |P'(z)|^2 z^0 \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1.$$

Now, by writing every polynomial in $\mathfrak{L}_n$ as $zP(z) \pm 1$ for some $P \in \mathfrak{L}_{n-1}$ and using the induction assumption, we have

$$\begin{aligned}
&\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} |P'(z)|^2 \, d\theta \\
&= \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} \left(|zP(z)+1|^2 + |zP(z)-1|^2 \right) \, d\theta \\
&= \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} \left(2\,|P(z)|^2 + 2 \right) d\theta \\
&= \frac{1}{2^{n}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} |P(z)|^2 \, d\theta + \frac{1}{2^{n}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} d\theta \\
&= n+1. \qquad \blacksquare
\end{aligned}$$

Lemma 3.2 *For an integer $m \geq 1$, we have*

$$\sum_{P' \in \mathfrak{L}_n} \int_0^{2\pi} |P'(z)|^2 z^m P'(z)^4 \, d\theta = 0.$$

Proof: We prove this result by induction on n. When $n = 0$, we have

$$\sum_{P' \in \mathfrak{L}_0} \int_0^{2\pi} |P'(z)|^2 z^m P'(z)^4 \, d\theta = 2 \int_0^{2\pi} z^m \, d\theta = 0,$$

because $\int_0^{2\pi} z^m \, d\theta = 0$ when $m \neq 0$. Now, by writing every polynomial in $\mathfrak{L}_n$ as $zP(z) \pm 1$ for some $P \in \mathfrak{L}_{n-1}$ and using the induction assumption, we have

$$\begin{aligned}
&\sum_{P' \in \mathfrak{L}_{k+1}} \int_0^{2\pi} \left|P'(z)\right|^2 z^m P'(z)^4 \, d\theta \\
&\quad = \sum_{P \in \mathfrak{L}_k} \int_0^{2\pi} z^m \Big[\left|zP(z)+1\right|^2 (zP(z)+1)^4 \\
&\qquad\qquad + \left|zP(z)-1\right|^2 (zP(z)-1)^4 \Big] \, d\theta.
\end{aligned}$$

We expand the integrand out and after some simple cancellation we get that the above expression is

$$\begin{aligned}
&\sum_{P \in \mathfrak{L}_k} \int_0^{2\pi} \Big[2\left|zP(z)\right|^2 z^{m+4} P(z)^4 + 20 \left|zP(z)\right|^2 z^{m+2} P(z)^2 \\
&\quad +10 \left|zP(z)\right|^2 z^m + 2z^m (5z^4 P(z)^4 + 10 z^2 P(z)^2 + 2) \Big] \, d\theta \\
&\quad = \sum_{P \in \mathfrak{L}_k} \int_0^{2\pi} 2 \left|zP(z)\right|^2 z^{m+4} P(z)^4 \, d\theta \\
&\qquad + \sum_{P \in \mathfrak{L}_k} \int_0^{2\pi} 20 \left|zP(z)\right|^2 z^{m+2} P(z)^2 \, d\theta \\
&\qquad + \sum_{P \in \mathfrak{L}_k} \int_0^{2\pi} 10 \left|zP(z)\right|^2 z^m \, d\theta
\end{aligned}$$

because $2z^m(5z^4P(z)^4 + 10z^2P(z)^2 + 2)$ is a polynomial with zero constant term in z.

Now the above integrals are equal to zero by Lemmas 2.2, 2.3 and the induction assumption. ■

Lemma 3.3 *For $P' \in \mathfrak{L}_n$, we have*

$$\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} |P'(z)|^2 \, P'(\overline{z})^2 \; d\theta \quad = \quad 3n + 1.$$

Proof: We write every polynomial in $\mathfrak{L}_n$ as $zP(z) \pm 1$ for some $P \in \mathfrak{L}_{n-1}$. Note that $\displaystyle\int_0^{2\pi} z^m \; d\theta = 0$ for $m \neq 0$. Now, using Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} |P'(z)|^2 \, P'(\overline{z})^2 \; d\theta \\
&= \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} \Big[|zP(z) + 1|^2 \left(\overline{z}P(\overline{z}) + 1\right)^2 \\
&\qquad + |zP(z) - 1|^2 \left(\overline{z}P(\overline{z}) - 1\right)^2 \Big] \; d\theta \\
&= \frac{1}{2^{n}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} \left[|P(z)|^2 \, z^{-2} P(\overline{z})^2 + 3\,|P(z)|^2 \right. \\
&\qquad \left. + 3z^{-2} P(\overline{z})^2 + 1 \right] \; d\theta \\
&= \frac{1}{2^{n}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} |P(z)|^2 \, z^{-2} P(\overline{z})^2 \; d\theta \\
&\qquad + \frac{1}{2^{n}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} 3\,|P(z)|^2 \; d\theta \\
&\qquad + \frac{1}{2^{n}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} 3z^{-2} P(\overline{z})^2 \; d\theta \\
&\qquad + \frac{1}{2^{n}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} d\theta \\
&= 0 + 3n + 3(0) + 1 \\
&= 3n + 1.
\end{aligned}$$

■

Lemma 3.4 *For an integer $m \geq 0$, we have*

$$\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} |P'(z)|^2 \, z^{2m} P'(\overline{z})^2 \, d\theta = \begin{cases} 3n+1 & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases} \tag{22}$$

Proof: Let $F_n(m)$ be the left hand side of Equation (22). By writing every polynomial in $\mathfrak{L}_n$ as $zP(z) \pm 1$, for some $P(z) \in \mathfrak{L}_{n-1}$, and using Lemma 3.1, and since $\overline{z} = z^{-1}$ and $\displaystyle\int_0^{2\pi} z^m \, d\theta = 0$ for $m \neq 0$, we have

$$\begin{aligned} F_n(m) &= \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} \Big[|zP(z)+1|^2 \left(\overline{z}P(\overline{z})+1\right)^2 \\ &\qquad + |zP(z)-1|^2 \left(\overline{z}P(\overline{z})-1\right)^2 \Big] d\theta \\ &= \frac{1}{2^n} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} \Big[|zP(z)|^2 \, \overline{z}^2 P(\overline{z})^2 + 3 \, |zP(z)|^2 \\ &\qquad + 3\overline{z}^2 P(\overline{z})^2 + 1 \Big] d\theta \\ &= \frac{1}{2^n} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} |P(z)|^2 \, z^{2(m-1)} P(\overline{z})^2 d\theta \\ &\qquad + \frac{1}{2^n} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} 3 \, |P(z)|^2 \, z^{2m} d\theta \\ &\qquad\quad + \frac{1}{2^n} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} 3 z^{2(m-1)} P(\overline{z})^2 d\theta \\ &\qquad\qquad + \frac{1}{2^n} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} d\theta \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^n}\sum_{P\in\mathfrak{L}_{n-1}}\frac{1}{2\pi}\int_0^{2\pi}|P(z)|^2\,z^{2(m-1)}P(\overline{z})^2d\theta+0\\
&\quad+\frac{1}{2^n}\sum_{P\in\mathfrak{L}_{n-1}}\frac{1}{2\pi}\int_0^{2\pi}3z^{2(m-1)}P(\overline{z})^2d\theta+0\\
&= F_{n-1}(m-1)+3C_{n-1}(m-1),
\end{aligned}
$$

where

$$
C_n(m):=\frac{1}{2^{n+1}}\sum_{P'\in\mathfrak{L}_n}\frac{1}{2\pi}\int_0^{2\pi}z^{2m}P'(\overline{z})^2\,d\theta=\begin{cases}1 & \text{if } m\le n,\\ 0 & \text{if } m>n,\end{cases}
$$

by Lemma 2.4. Clearly,

$$
F_0(0)=1 \text{ and } F_0(m)=\int_0^{2\pi}z^{2m}\,d\theta=0 \text{ for } m\ge 1.
$$

For $n\ge 1$, we have

$$
F_n(m)=F_{n-1}(m-1)+3C_{n-1}(m-1).
$$

If $m>n$, then $C_{n-1}(m-1)=0$ and hence

$$
F_n(m)=F_{n-1}(m-1)=\cdots=F_1(m-n+1)=F_0(m-n)=0.
$$

If $m\le n$, then

$$
\begin{aligned}
F_n(m) &= F_{n-1}(m-1)+3C_{n-1}(m-1)\\
&= F_{n-2}(m-2)+3C_{n-2}(m-2)+3C_{n-1}(m-1)\\
&= F_{n-3}(m-3)+3C_{n-3}(m-3)+3C_{n-2}(m-2)\\
&\qquad+3C_{n-1}(m-1)\\
&\;\vdots\\
&= F_{n-m+1}(1)+3C_{n-m+1}(1)+3C_{n-m+2}(2)+\dots\\
&\quad+3C_{n-3}(m-3)+3C_{n-2}(m-2)+3C_{n-1}(m-1)
\end{aligned}
$$

$$\begin{aligned}
&= F_{n-m}(0) + 3C_{n-m}(0) + 3C_{n-m+1}(1) + 3C_{n-m+2}(2) \\
&\quad + \ldots + 3C_{n-3}(m-3) + 3C_{n-2}(m-2) + 3C_{n-1}(m-1) \\
&= F_{n-m}(0) + 3\left(\underbrace{1+1+\ldots+1}_{m\ times}\right) \\
&= F_{n-m}(0) + 3m \\
&= [3(n-m)+1] + 3m \\
&= 3n+1. \qquad \blacksquare
\end{aligned}$$

Lemma 3.5 *For an integer $m \geq 1$, we have*

$$\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} |P'(z)|^6 z^{2m}\ d\theta$$

$$= \begin{cases} \frac{9}{2}(n-m)^2 + \frac{15}{2}(n-m) & \\ \qquad +\frac{3}{2}(n-m)(3n+3m-1) + 9m - 3 & \text{if } m \leq n \\ 0 & \text{if } m > n. \end{cases}$$

Proof: Let

$$D_n(m) := \frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} |P'(z)|^6 z^{2m}\ d\theta.$$

Then

$$\begin{aligned} D_n(m) = \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} &\left(|zP(z)+1|^6\right. \\ &\left. + |zP(z)-1|^6\right) z^{2m}\ d\theta. \end{aligned}$$

We expand the integrand out and use Lemmas 2.5, 2.4, and 3.4. We derived that for $n, m \geq 1$,

$$D_n(m) = D_{n-1}(m) + 9B_{n-1}(m) + 3C_{n-1}(m-1) + 3F_{n-1}(m-1).$$

It is clear that $D_0(m) = 0$. Thus if $m > n \geq 1$, then $B_{n-1}(m)$, $C_{n-1}(m-1)$, and $F_{n-1}(m-1)$ are all equal to zero and hence

$$D_n(m) = D_{n-1}(m) = D_{n-2}(m) = \cdots = D_0(m) = 0.$$

If $1 \leq m \leq n$, then

$$\begin{aligned}
D_n(m) &= D_{n-1}(m) + 9B_{n-1}(m) + 3C_{n-1}(m-1) \\
&\qquad +3F_{n-1}(m-1) \\
&= D_{n-2}(m) + 9B_{n-2}(m) + 9B_{n-1}(m) \\
&\qquad +3C_{n-2}(m-1) + 3C_{n-1}(m-1) \\
&\qquad\quad +3F_{n-2}(m-1) + 3F_{n-1}(m-1) \\
&\vdots \\
&= D_m(m) + 9B_m(m) + 9B_{m+1}(m) + \ldots \\
&\quad +9B_{n-1}(m) + 3C_m(m-1) + 3C_{m+1}(m-1) + \\
&\qquad \ldots + 3C_{n-1}(m-1) + 3F_m(m-1) \\
&\quad +3F_{m+1}(m-1) + \ldots + 3F_{n-1}(m-1) \\
&= (9m-3) + 9\sum_{j=m}^{n-1} B_j(m) + 3\sum_{j=m}^{n-1} C_j(m-1) \\
&\qquad +3\sum_{j=m}^{n-1} F_j(m-1) \\
&= (9m-3) + 9\sum_{j=m}^{n-1}(j-m+1) + 3\sum_{j=m}^{n-1} 1 \\
&\qquad +3\sum_{j=m}^{n-1}(3j+1) \\
&= (9m-3) + 9\left[\frac{1}{2}(n-m)^2 + \frac{1}{2}(n-m)\right] \\
&\qquad +3(n-m) + \frac{3}{2}(n-m)(3n+3m-1)
\end{aligned}$$

$$= \frac{9}{2}(n-m)^2 + \frac{15}{2}(n-m) + \frac{3}{2}(n-m)(3n+3m-1) + 9m - 3. \qquad \blacksquare$$

Lemma 3.6 *For an integer $m \geq 1$, we have*

$$\frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} |P'(z)|^6 z^{2m} P'(z)^2 d\theta$$

$$= \begin{cases} \frac{15}{4}(n-m)^3 + \frac{75}{8}(n-m)^2 + \frac{21}{4}(n-m) \\ \quad +\frac{5}{8}(9n+3m-2)(n-m) \\ \quad +\frac{15}{4}(5n+m-6)(n-m)(n-m+2) \\ \quad +\frac{15}{2}(n+m) - 7 & \textit{if } n+m \equiv 0 \pmod 2 \\ & \textit{and } m \leq n \\ \\ \frac{15}{4}(n-m)^3 + \frac{75}{8}(n-m)^2 + \frac{51}{4}(n-m) \\ \quad +\frac{5}{8}(n-m-1)(9n+3m+1) \\ \quad +\frac{15}{4}(n-m-1) \\ \quad (5n^2 - 4nm + 9n - m^2 + 9m - 12) \\ \quad +\frac{105}{2}(n+m) - \frac{563}{8} & \textit{if } n+m \equiv 1 \pmod 2 \\ & \textit{and } m \leq n, \\ \\ 0 & \textit{if } m > n. \end{cases}$$

Proof: Let

$$E_n(m) := \frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} |P'(z)|^6 z^{2m} P'(z)^2 d\theta.$$

Then

$$\begin{aligned} E_n(m) &= \frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} \left\{ |zP(z)+1|^6 (zP(z)+1)^2 \right. \\ &\qquad \left. + |zP(z)-1|^6 (zP(z)-1)^2 \right\} d\theta. \end{aligned}$$

We expand the integrand out and use Lemmas 2.4, 2.5, 3.4 and 3.5. We derived that for $n, m \geq 1$,

$$\begin{aligned} E_n(m) &= E_{n-1}(m+1) + 10D_{n-1}(m) + 15A_{n-1}(m+1) \\ &\qquad +30B_{n-1}(m) + 5F_{n-1}(m-1) + 3C_{n-1}(m-1), \end{aligned}$$

where

$$\begin{aligned} A_n(m) &:= \frac{1}{2^{n+1}} \sum_{P' \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} |P'(z)|^4 z^{2m} P'(z)^2 \, d\theta \\ &= \begin{cases} \frac{3}{2}(n-m)^2 + \frac{7}{2}(n-m) + \frac{3-(-1)^{m+n}}{2} & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases} \end{aligned}$$

by Lemma 2.6. It is clear that $E_0(m) = 0$. Thus if $m > n \geq 1$, then

$$\begin{aligned} D_{n-1}(m) &= A_{n-1}(m+1) = B_{n-1}(m) \\ &= F_{n-1}(m-1) = C_{n-1}(m-1) = 0 \end{aligned}$$

and hence

$$\begin{aligned} E_n(m) &= E_{n-1}(m+1) = E_{n-2}(m+2) = \cdots \\ &= E_1(m+n-1) = E_0(m+n) = 0. \end{aligned}$$

Suppose $1 \leq m \leq n$. If $n + m \equiv 0 \pmod 2$, then

$$
\begin{aligned}
&E_n(m)\\
= \;& E_{\frac{n+m}{2}}\left(\frac{n+m}{2}\right) + 10\Bigg[D_{n-1}(m) + D_{n-2}(m+1)\\
&+D_{n-3}(m+2) + \ldots + D_{\frac{n+m}{2}}\left(\frac{n+m}{2} - 1\right)\Bigg]\\
&+15\Bigg[A_{n-1}(m+1) + A_{n-2}(m+2) + \ldots + A_{\frac{n+m}{2}}\left(\frac{n+m}{2}\right)\Bigg]\\
&+30\Bigg[B_{n-1}(m) + B_{n-2}(m+1) + \ldots + B_{\frac{n+m}{2}}\left(\frac{n+m}{2} - 1\right)\Bigg]\\
&+5\Bigg[F_{n-1}(m-1) + F_{n-2}(m) + \ldots + F_{\frac{n+m}{2}}\left(\frac{n+m}{2} - 2\right)\Bigg]\\
&+3\Bigg[C_{n-1}(m-1) + C_{n-2}(m) + \ldots + C_{\frac{n+m}{2}}\left(\frac{n+m}{2} - 2\right)\Bigg].
\end{aligned}
$$

Using Lemmas 2.4, 2.5, 2.6, 3.4 and 3.5, we see that

$$
\begin{aligned}
&E_n(m)\\
= \;& \tfrac{15}{2}(n+m) - 7 + 10\left[\tfrac{3}{8}\left(5n+m-6\right)(n-m)(n-m+2)\right]\\
&+15\left[\tfrac{1}{4}(n-m)^3 + \tfrac{1}{8}(n-m)^2 - \tfrac{3}{4}(n-m)\right]\\
&\quad +30\left[\frac{(n-m)^2}{4} + \frac{n-m}{2}\right]\\
&\qquad +5\left[\tfrac{1}{8}\left(9n+3m-2\right)(n-m)\right] + \tfrac{3}{2}(n-m)\\
= \;& \tfrac{15}{4}(n-m)^3 + \tfrac{75}{8}(n-m)^2 + \tfrac{21}{4}(n-m)\\
&+\tfrac{5}{8}(9n+3m-2)(n-m)\\
&\quad +\tfrac{15}{4}(5n+m-6)(n-m)(n-m+2)\\
&\qquad +\tfrac{15}{2}(n+m) - 7.
\end{aligned}
$$

The case $n+m \equiv 1 \pmod 2$ can be proved in the same way. ■

In particular, $E_0(1) = 0$ and for $n \geq 1$

$$E_n(1) = \frac{45}{2}n^3 - 15n^2 - 11n + \frac{19}{2} + \frac{1}{2}(-1)^n(15n - 19). \quad (23)$$

We now come to the proof of $\mu_n(10)$. Since

$$\begin{aligned}
&|zP(z)+1|^{10} + |zP(z)-1|^{10} \\
&= 2\,|zP(z)|^{10} + 50\,|zP(z)|^8 + 200\,|zP(z)|^6 + 200\,|zP(z)|^4 \\
&\quad +50\,|zP(z)|^2 + 2 + 20\,|zP(z)|^6\,(z^2P(z)^2 + \overline{z}^2P(\overline{z})^2) \\
&\quad +100\,|zP(z)|^4\,(z^2P(z)^2 + \overline{z}^2P(\overline{z})^2) \\
&\quad +100\,|zP(z)|^2\,(z^2P(z)^2 + \overline{z}^2P(\overline{z})^2) \\
&\quad +10\,|zP(z)|^2\,(z^4P(z)^4 + \overline{z}^4P(\overline{z})^4) \\
&\quad +10(z^4P(z)^4 + \overline{z}^4P(\overline{z})^4) + 20(z^2P(z)^2 + \overline{z}^2P(\overline{z})^2),
\end{aligned}$$

it follows from Lemmas 2.6, 3.6 and (23) that

$$\begin{aligned}
\mu_n(10) &= \mu_{n-1}(10) + 25\mu_{n-1}(8) + 100\mu_{n-1}(6) \\
&\quad +100\mu_{n-1}(4) + 25\mu_{n-1}(2) + 1 + 20E_{n-1}(1) \\
&\quad +100A_{n-1}(1).
\end{aligned}$$

In view of Theorem 2.7 and (23), we have

$$\begin{aligned}
&\mu_n(10) \\
&= \mu_{n-1}(10) + 25(24n^4 - 66n^3 + 58n^2 - 9n - 3 + 3(-1)^n) \\
&\quad +100(6n^3 - 9n^2 + 4n) + 100(2n^2 - n) + 25n + 1 \\
&\quad +20\left[\tfrac{45}{2}n^3 - \tfrac{165}{2}n^2 + \tfrac{173}{2}n - 17 - \tfrac{1}{2}(-1)^n\,(15n - 34)\right] \\
&\quad +100\left[\tfrac{3}{2}(n-2)^2 + \tfrac{7}{2}(n-2) + \frac{3-(-1)^n}{2}\right] \\
&= 120n^5 + 150n^4 - 350n^3 + 265n^2 \\
&\qquad +281n - 144 - 5(-1)^n(15n - 29). \qquad \blacksquare
\end{aligned}$$

Thus we have proved

Theorem 3.7 *For any integer $n \geq 0$,*

$$\begin{aligned}\mu_n(10) &= 120n^5 + 150n^4 - 350n^3 + 265n^2 \\ &\quad +281n - 144 - 5(-1)^n(15n - 29).\end{aligned}$$

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