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The Exact Gossiping Problem for $k \geq 8$ Messages

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Abstract: This study of the exact gossiping problem extended the results for $k \geq 8$ messages. In generating the minimum number of call sequence, a step by step process was created to generate $E(n,k)$, the minimum number of call sequence where n is the number of vertices and k the number of messages, which consequently produced the initial function. Three Lemmas were presented. These lemmas were used to prove the theorem for the exact gossiping problem for $k \geq 8$ messages. The resulting function used in the theorem is given below

$$\begin{aligned} E(n,k) &= E(km+r,k) \\ &\leq \begin{cases} 2km-4m+2r-4 & \text{, if } r=k \\ 2km-4m+k+r-4 & \text{, if } k<r<2k-1 \\ 2km-4m+3k-5 & \text{, if } r=2k-1 \end{cases} \end{aligned}$$

where $k \geq 8$, $m \geq 0$ and $k \geq r \geq 2k-1$.

1 Introduction

Gossiping is considered as one of the basic communication patterns. Gossiping, also known as complete exchange and all-to-all communication, is a communication problem in which all processor has a unique message to be transmitted to every other processor.

Graphs are discrete structures consisting of vertices and edges that connect these vertices. A graph model as defined in the field of mathematics, is a graph $G = (V, E)$ consisting of $V(G)$, a nonempty set of vertices (or nodes) and $E(G)$, a set of edges. Each of these edges has either one or two vertices associated with it called its endpoints (Rosen, 2008).

Graph theory only deals with relationships between explicitly defined elements which are limited in number in order to determine certain traditional properties of graphs, such as the shortest paths, the Hamiltonian cycle, etc. (Mathis, 2007).

The basic gossiping problem is defined as follows: *there are n individuals, each individual having an item of gossip or a message. The goal is to transmit each item of gossip or messages to a number of individual included in the set of nodes.*

The communication between these nodes typically proceeds in rounds, with the objective of minimizing the number of call succession in each round. The gossiping process advances sequentially across the nodes in the graph relaying the message/s fully across the other.

The exact gossiping problem deals with minimizing the number of call sequence $E(n, k)$ for n nodes to obtain exactly k numbers of messages. This could be best modeled with a multigraph (graph with multiple edges, without loops, connecting the same vertices).

Tsay and Chang in 1995 generated a method in solving exact gossiping problems for particular numbers of messages. Their study gave out solutions for when $k \leq 4$. Paredes in 2001 conducted a study that will minimize $E(n, k)$ for when $k = 5$ and $n \geq 5$. Paderanga in 2003 conducted a study that will minimize $E(n,k)$ for when $k = 6$ and $n \geq 6$. Salvador in 2008 conducted a study that will minimize $E(n,k)$ for when $k = 7$ and $n \geq 7$. All the above mentioned utilized the methods of Tsay and Chang in their study. In this study, an optimization method, particularly the minimization of the call sequences, was used. This method was directed by the principles of Graph theory particularly the method used by Tsay and Chang in developing the solutions for the exact gossiping problem for $0 \leq k \leq 4$.

2 Materials and Methods

Graph theory took the form of optimization in terms of minimizing the call sequences in the gossiping process.

A set of n vertices (where $n \geq k$) was considered such that each of these vertices correspond to a node or an individual. Each of these vertices had a unique piece of information/gossip item. No connection between any vertices had been established at the beginning of the procedure.

When two vertices were finally joined together by an edge (possibly curved), this edge was counted as the first call sequence C_1, the second as C_2 and so on. Each call sequence occurred such that full exchange of gossip information was made per call sequence. The subscript of this call sequence indicated the order of the call made between each vertex therefore the order of the call mattered significantly. Messages were distributed among all the n vertices until all n vertices received exactly k messages each.

The process to generate the minimum number of call sequence $E(n,k)$ was directed by the general guiding rule from Tsay and Chang's definition of the exact gossiping problem. This step by step process was obtained to generate the general solutions of $E(n,k)$ for all values of $k \geq 8$ and

$k < n < 2k - 1$. It also returned the minimum number of call sequence $E(n,k)$ as well as a graph for each $E(n,k)$.

2.1 Assumptions

1. In the graph $G(V,E)$, the finite set of vertices $V = \{v_1, v_2, v_3, ...\}$ corresponds to n nodes/persons involved in the gossip and the finite set of edges $E = \{e_1, e_2, e_3, ...\}$ corresponds to the number of call sequences.

2. Each vertex contains a unique information/gossip to be distributed later on in the gossiping process.

3. When any of the two vertices are connected by an edge, the information each vertex has will be faithfully exchanged.

3 Results

In this section, the results of the study were presented together with their proofs. A step by step process for generating the call sequence $E(n,k)$ where $k \geq 8$ and $k \leq n \leq 2k - 1$, is presented in Lemma 2 along with the function for $E(n,k)$ where $k \geq 8$ and $k \leq n \leq 2k - 1$, which was consequently generated with the step by step process.

Three lemmas were used to aid to the proof of Theorem 1. The first lemma was derived from the study of $Tsay$ and $Chang$ and was also used in proving Lemma 3. Lemma 2 generated the initial function which was used for Lemma 3. Lemma 3 expressed n as a component combination of $km + r$, where $k \geq 8$, $m \geq 0$ and $k \leq r \leq 2k - 1$ where c , the number of call sequence of $E(n,k)$, is optimal. Theorem 1 presented the general function for $E(n,k)$ for $k \geq 8$ and $n \geq k$.

The following definitions are helpful in the proofs to follow:

Definition 3.1 A graph G is a finite nonempty set $V(G)$ of objects called nodes and a set $E(G)$ of 2-element subsets of $V(G)$ called edges. The set $V(G)$ is called the vertex of set G and $E(G)$ its edge set. The vertices of a graph can be represented by points, which will then be connected by a line each time corresponding pairs of vertices form an angle.

Definition 3.2 Let G be a multigraph of n vertices and c is a call sequence for G. A call sequence c is optimal if every vertex of G has exactly k messages.

Definition 3.3 A subgraph H of G is a graph whose points and lines are also in G so that, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 3.4 A connected component (or simply a component) of H of an undirected graph is a minimal connected subgraph. Maximal means that it is the largest possible subgraph.

Definition 3.5 The cardinality of G, denoted by $|G|$, is the number of call sequences in G.

Definition 3.6 Let G be a multigraph of n vertices. A component combination of G are all possible connected maximal subgraphs $G_1, G_2, \ldots, G_m$ of G that has an optimal call sequence c and $|G| = |G_1| + |G_2| + \cdots + |G_m|$.

Definition 3.7 A tree is a connected graph with no cycles.

The previous study of $Tsay$ and $Chang$ (1995) gave the initial results to help us prove the solution for $E(n, k)$ where $k \geq 8$. This is given in Lemma 1. The proof of this result was already presented in their paper.

Lemma 3.8 *(Tsay and Chang, 1995).* $E(m+n, k) \leq E(m, k) + E(n, k)$.

Lemma 3.9 *For* $k \leq n \leq 2k-1$

$$E(n,k) = E(km+r,k) = \begin{cases} 2n-4, & \text{if} \quad n = k \\ k+n-4, & \text{if} \quad k < n < 2k-1 \\ 3k-5, & \text{if} \quad n = 2k-1 \end{cases}$$

Lemma 3.10 *Let* G *be a multigraph of* n *vertices, and* c *is a call sequence on* G*. If* G *is a component combination of* $km + r$*, where* $k \geq 8$*,* $m \geq 0$ *and* $k \leq r \leq 2k-1$ *components, then* c *is optimal.*

Theorem 3.11 *For* $n \geq k$*,* n *can be expressed as* $n = km + r$*, for all* $k \geq 8$*,* $m \geq 0$ *and* $k \leq r \leq 2k-1$*.*

$$\begin{aligned} E(n,k) &= E(km+r,k) \\ &\leq \begin{cases} 2km - 4m + 2r - 4 & , \text{ if } r = k \\ 2km - 4m + k + r - 4 & , \text{ if } k < r < 2k-1 \\ 2km - 4m + 3k - 5 & , \text{ if } r = 2k-1 \end{cases} \end{aligned}$$

Proof: By Lemma 3.9, the results are proven true for $k < r < 2k-1$. The following cases are to be considered:

Case 1 : $r = k$

$$\begin{aligned} E(n,k) &= E(km+r,k) \\ &\leq E(km,k) + E(r,k), && \text{by Lemma 1.} \\ &\leq [E(\overbrace{k+k+k+\cdots+k}^{m \text{ copies of } k}, k)] + E(r,k), && \text{Lemma 1.} \\ &\leq [(\overbrace{E(k,k)+E(k,k)+E(k,k)+\cdots+E(k,k)}^{m \text{ copies of } E(k,k)}, k)] \\ &\quad + E(r,k), && \text{by Lemma 1.} \end{aligned}$$

$$
\begin{aligned}
&= mE(k,k) + E(r,k) \\
&= m(2k-4) + (2r-4), && \text{by Lemma 2.} \\
&= 2km - 4m + 2r - 4.
\end{aligned}
$$

Case 2 : $k < r < 2k-1$

$$
\begin{aligned}
E(n,k) &= E(km+r,k) \\
&\leq E(km,k) + E(r,k), && \text{by Lemma 1.} \\
&\leq [E(\overbrace{k+k+k+\cdots+k}^{m\ copies\ of\ k},k)] + E(r,k), && \text{by Lemma 1.} \\
&\leq [(\overbrace{E(k,k)+E(k,k)+E(k,k)+\cdots+E(k,k)}^{m\ copies\ of\ E(k,k)},k)] \\
&\quad + E(r,k), && \text{by Lemma 1.} \\
&= mE(k,k) + E(r,k) \\
&= m(2k-4) + (2r-4), && \text{by Lemma 2.} \\
&= 2km - 4m + k + r - 4.
\end{aligned}
$$

Case 3 : $r = 2k-1$

$$
\begin{aligned}
E(n,k) &= E(km+r,k) \\
&\leq E(km,k) + E(r,k), && \text{by Lemma 1.} \\
&\leq [E(\overbrace{k+k+k+\cdots+k}^{m\ copies\ of\ k},k)] + E(r,k), && \text{by Lemma 1.} \\
&\leq [(\overbrace{E(k,k)+E(k,k)+E(k,k)+\cdots+E(k,k)}^{m\ copies\ of\ E(k,k)},k)] \\
&\quad + E(r,k), && \text{by Lemma 1.} \\
&= mE(k,k) + E(r,k) \\
&= m(2k-4) + (3k-5), && \text{by Lemma 2.} \\
&= 2km - 4m + 3k - 5.
\end{aligned}
$$

Hence,

$$\begin{aligned} E(n,k) &= E(km+r,k) \\ &\leq \begin{cases} 2km-4m+2r-4 & \text{, if } r=k \\ 2km-4m+k+r-4 & \text{, if } k<r<2k-1 \\ 2km-4m+3k-5 & \text{, if } r=2k-1. \end{cases} \end{aligned}$$

4 Conclusion

The process of exhaustion/inspection was used to generate a step by step method of generating the graphs as well as the initial values which were used to come up with initial values for $E(n,k)$ for $k \geq 8$ and $k \leq n \leq 2k-1$, and consequently an initial function. The initial function generates the values of $E(n,k)$ for $k \leq n \leq 2k-1$. This function was used to obtain the general solution which can be expressed as the function given below,

$$\begin{aligned} E(n,k) &= E(km+r,k) \\ &\leq \begin{cases} 2km-4m+2r-4 & \text{, if } r=k \\ 2km-4m+k+r-4 & \text{, if } k<r<2k-1 \\ 2km-4m+3k-5 & \text{, if } r=2k-1. \end{cases} \end{aligned}$$

where $k \geq 8$, $m \geq 0$ and $k \leq r \leq 2k-1$. The result was proven mathematically.

As a variant to the many graph theory problems, the exact gossiping problem can be considered as a communication and networking problem. Knowing the importance of communication, this study could cater to the development and advancement of the networking technology. Also, the result of this study may contribute to the progress in the real world communication networking situation as this study considers networks with nodes ≥ 8.

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