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# Notes on Pell and Pell-Lucas Sequences with Negative Subcripts

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**Abstract**

In this study, we establish some properties of Pell and Pell-Lucas sequences with negative subcripts by using $nth$ power of a special matrix. Some of the properties for these sequences are obtained by matrix algebra.

## 1 Introduction

Pell and Pell-Lucas numbers offer opportunities for experimentation, conjecture, and problem-solving techniques, connecting the fields of analysis, geometry, trigonometry, and various areas of discrete mathematics, number theory, graph theory, linear algebra, and combinatorics. Pell and Pell-Lucas numbers have extracting numerous interesting properties. Therefore, there are many papers about Pell and Pell-Lucas numbers in the last decade years. As with Pell's equation, the name of the Pell numbers stems from Leonhard Euler's mistaken attribution of the equation and the numbers derived from it to John Pell. The Pell-Lucas numbers are also named after Édouard Lucas, who studied sequences defined by recurrences of this type.

Matrix algebra has very important use for the theory of special integer sequences. Hence, in [5], Williams studied any power of a matrix of the type $2 \times 2$. Bergum and Hoggatt investigated the sums and products for recurring sequences in [3]. Laughlin denoted some combinatorial features obtained by any power of some matrices in [3, 4]. Then, Belbachir found linear recurrent sequences and powers of a square matrix in [6]. The authors demonstrated some combinatorial properties by determinant and trace of any power of a given matrix whose entries are generalized Fibonacci and Lucas numbers in [10]. Halici, Akyuz studied Fibonacci and Lucas sequences at negative indices in [8]. Dasdemir investigated Mersenne, Jacobsthal and Jacobsthal-Lucas sequences with negative indices in [14]. Uygun examined some properties of the Jacobsthal sequence at negative subcripts in [15].

The Pell numbers $\rho_n$ are terms of the sequence $\{1, 2, 5, 12, 29, 70, ...\}$ denoted by the following recurrence relation

$$\rho_{n+2} = 2\rho_{n+1} + \rho_n$$

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for any natural numbers beginning with the values $\rho_0 = 0,\ \rho_1 = 1$. Similarly, the Pell-Lucas numbers $\sigma_n$ are terms of the sequence $\{2, 2, 6, 14, 34, 82, 198, ...\}$ denoted by the following recurrence relation

$$\sigma_{n+2} = 2\sigma_{n+1} + \sigma_n$$

for any natural numbers beginning by the values $\sigma_0 = 2,\ \sigma_1 = 2$ in [1]. We can learn more information about Pell and Pell-Lucas numbers and their generalizations in [1, 2, 7-9, 11, 13, 14]. The relation between these sequences are given as

$$\begin{aligned} \sigma_n &= \rho_{n-1} + \rho_{n+1}, \\ 8\rho_n &= \sigma_{n-1} + \sigma_{n+1}. \end{aligned}$$

Binet formula enables us to state Pell and Pell-Lucas numbers easily. It can be clearly obtained from the roots $r_1 = 1+\sqrt{2}$ and $r_2 = 1-\sqrt{2}$ of characteristic equation of the recurrence relation as the form $x^2 = x + 2$. The Pell and Pell-Lucas numbers have the Binet formulas as

$$\begin{aligned} \rho_n &= \frac{r_1^n - r_2^n}{r_1 - r_2}, \\ \sigma_n &= r_1^n + r_2^n. \end{aligned}$$

Pell and Pell-Lucas numbers at negative indices are defined by using the following equalities:

$$\begin{aligned} \rho_{-n} &= (-1)^{n+1}\rho_n, \quad (1) \\ \sigma_{-n} &= (-1)^n \sigma_n. \quad (2) \end{aligned}$$

The first Pell numbers at negative indices are $\rho_{-1} = 1$, $\rho_{-2} = -2$, $\rho_{-3} = 5$, $\rho_{-4} = -12$, $\rho_{-5} = 29$, $\rho_{-6} = -70$. Similarly, the first Pell-Lucas numbers at negative indices are $\sigma_{-1} = -2$, $\sigma_{-2} = 6$, $\sigma_{-3} = -14$, $\sigma_{-4} = 34$, $\sigma_{-5} = -82$, $\sigma_{-6} = 198$. The relation between the sequences with negative indices are given as

$$\begin{aligned} \rho_{-(n+1)} + \rho_{-(n-1)} &= \sigma_{-n}, \quad (3) \\ \sigma_{-(n+1)} + \sigma_{-(n-1)} &= 8\rho_{-n}. \quad (4) \end{aligned}$$

In [5], Williams gave a well-known formula that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^n = \begin{cases} \frac{x_1^n - x_2^n}{x_1 - x_2} A - \frac{x_1^{n-1} - x_2^{n-1}}{x_1 - x_2} I_2, & x_1 \neq x_2 \\ n x^{n-1} A - (n-1)\det(A) x^{n-2} I_2, & x_1 = x_2 = x \end{cases}$$

where $x_1$, $x_2$ are the eigenvalues of the characteristic equation of the matrix $A$

$$r^2 - (a+d)r + \det(A) = 0.$$

Laughlin, in [3, 4] gave that if $A$ is a $2 \times 2$ matrix as $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the $nth$ power of $A$ is given by

$$A^n = \begin{bmatrix} x_n - d x_{n-1} & b x_{n-1} \\ c x_{n-1} & x_n - a x_{n-1} \end{bmatrix} \quad (5)$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} n-i \\ i \end{pmatrix} T^{n-2i}(-D)^i$, $T =$trace of $A$, $D =$determinant of $A$.

**Proposition 1.1.** *(Catalan's identity)*
*The following properties for the Pell and Pell-Lucas numbers at negative indices are valid:*

$$\begin{aligned} \rho_{-(n+r)}\rho_{-(n-r)} - \rho_{-n}^2 &= (-1)^{n-r+1}, \\ \sigma_{-(n+r)}\sigma_{-(n-r)} - \sigma_{-n}^2 &= 8(-1)^{n-r}\rho_r^2. \end{aligned}$$

**Proposition 1.2.** *(Simpson Property)*
*The properties for the Pell and Pell-Lucas numbers at negative indices are obtained by the above proposition:*

$$\rho_{-(n+1)}\rho_{-(n-1)} - \rho_{-n}^2 = (-1)^n, \tag{6}$$

$$\sigma_{-(n+1)}\sigma_{-(n-1)} - \sigma_{-n}^2 = 8(-1)^{n-1}. \tag{7}$$

**Proposition 1.3.** *(D'Ocagne Property)*
*The following properties for the Pell and Pell-Lucas numbers at negative indices are valid:*

$$\rho_{-(m+1)}\rho_{-n} - \rho_{-m}\rho_{-(n+1)} = (-1)^{n+1}\rho_{-(m-n)}, \tag{8}$$

$$\sigma_{-(m+1)}\sigma_{-n} - \sigma_{-m}\sigma_{-(n+1)} = 8(-1)^{n}\rho_{-(m-n)}, \tag{9}$$

$$\sigma_{-m}\rho_{-(n-1)} - \sigma_{-(m-1)}\rho_{-n} = (-1)^{n}\sigma_{-(m-n)}, \tag{10}$$

$$\rho_{-m}\sigma_{-(n-1)} - \rho_{-(m-1)}\sigma_{-n} = (-1)^{n+1}\sigma_{-(n-m)}. \tag{11}$$

*Proof.* The proofs are similar so we only investigate one of them. For example, for (1), we use (1, 2) and we have

$$\rho_{-m}\sigma_{-(n-1)} - \rho_{-(m-1)}\sigma_{-n} = (-1)^{m+n}\left(\rho_m\sigma_{(n-1)} - \rho_{(m-1)}\sigma_n\right).$$

By the Binet formulas of the Pell and Pell-Lucas numbers, we have

$$\begin{aligned} (-1)^{m+n}\left(\rho_m\sigma_{(n-1)} - \rho_{(m-1)}\sigma_n\right) &= \frac{r_1^m - r_2^m}{r_1 - r_2}\left(r_1^{n-1} + r_2^{n-1}\right) - \frac{r_1^{m-1} - r_2^{m-1}}{r_1 - r_2}\left(r_1^n + r_2^n\right) \\ &= \frac{r_1^m r_2^{n-1} - r_2^m r_1^{n-1} - r_1^{m-1} r_2^n + r_1^n r_2^{m-1}}{r_1 - r_2} \\ &= \frac{-r_1^m r_2^n\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - r_1^n r_2^m\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}{r_1 - r_2} \\ &= -r_1^m r_2^n - r_1^n r_2^m \\ &= (-1)^{n+1}\left(r_1^{m-n} + r_2^{m-n}\right) \\ &= (-1)^{n+1}\sigma_{m-n} \end{aligned}$$

□

**Proposition 1.4.** *(Tagiuri's identitiy)*
*Pell and Pell-Lucas numbers at negative indices have the following features:*

$$\begin{aligned} \rho_{-(m+k)}\rho_{-(n-k)} - \rho_{-m}\rho_{-n} &= (-1)^{n+k-1}\rho_{-k}\rho_{-(m-n+k)}, \\ \sigma_{-(m+k)}\sigma_{-(n-k)} - \sigma_{-m}\sigma_{-n} &= 2\sqrt{2}(-1)^{n+k-1}\sigma_{-(m-n+k)}. \end{aligned}$$

**Proposition 1.5.** *(Honsberger's identity)*
*Pell and Pell-Lucas numbers at negative indices hold the identities:*

$$\begin{aligned} \rho_{-(m+1)}\rho_{-(n+1)} + \rho_{-m}\rho_{-n} &= (-1)^{m+n}(\sigma_{-(m+n+2)} + \sigma_{-(m+n)})/8, \\ \sigma_{-(m+1)}\sigma_{-(n+1)} + \sigma_{-m}\sigma_{-n} &= (-1)^{m+n}(\sigma_{-(m+n+2)} + \sigma_{-(m+n)}). \end{aligned}$$

## 2 Main Results

**Theorem 2.1.** *Let us consider a special matrix as*

$$P = \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}. \tag{12}$$

*The nth power of P is calculated by using the Pell sequence and the Pell-Lucas sequence at negative indices as*

$$P^n = \begin{cases} 8^{\frac{n}{2}} \begin{bmatrix} \rho_{-(n+1)} & \rho_{-n} \\ \rho_{-n} & \rho_{-(n-1)} \end{bmatrix}, & \text{if } n \text{ is even} \\ 8^{\frac{n-1}{2}} \begin{bmatrix} \sigma_{-(n+1)} & \sigma_{-n} \\ \sigma_{-n} & \sigma_{-(n-1)} \end{bmatrix}, & \text{if } n \text{ is odd} \end{cases} \tag{13}$$

*Proof.* For $n = 1, 2$ the statement is valid. Let us it is true for all $k \leq n$, and $n$ is even. Then for $n = k + 1$, we investigate the validity of the claim.

$$P^{n+1} = P^n P = 8^{\frac{n}{2}} \begin{bmatrix} \rho_{-(n+1)} & \rho_{-n} \\ \rho_{-n} & \rho_{-(n-1)} \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}.$$

For the (1,1)-th element of $P^{n+1}$, by (1, 2, 4), we get

$$\begin{aligned} 8^{\frac{n}{2}}[6(-1)^{n+2}\rho_{n+1} - 2(-1)^{n+1}\rho_n] &= 8^{\frac{n}{2}}[(-1)^{n+2}(6\rho_{n+1} + 2\rho_n)] \\ &= 8^{\frac{n}{2}}[(-1)^{n+2}\sigma_{n+2}] \\ &= 8^{\frac{n}{2}}(\sigma_{-(n+2)}). \end{aligned}$$

The other elements are obtained similarly. Assume that $n$ is odd, then we get

$$P^{n+1} = 8^{\frac{n-1}{2}} \begin{bmatrix} \sigma_{-(n+1)} & \sigma_{-n} \\ \sigma_{-n} & \sigma_{-(n-1)} \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}.$$

For the (1,1)-th element of $P^{n+1}$, by (1, 2, 5), we get

$$\begin{aligned} 8^{\frac{n-1}{2}}[6(-1)^{n+1}\sigma_{n+1} - 2(-1)^{n}\sigma_n] &= 8^{\frac{n-1}{2}}[(-1)^{n+1}(6\sigma_{n+1} + 2\sigma_n)] \\ &= 8^{\frac{n+1}{2}}[(-1)^{n+1}\rho_{n+2}]. \end{aligned}$$

The other elements are also obtained similarly. □

**Theorem 2.2.** *For positive integers n, the explicit closed form expressions for the Pell and Pell-Lucas sequences at negative indices are evaluated as*

$$\begin{aligned} \rho_{-n} &= \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} (-1)^{i+1} \frac{8^{\frac{n}{2}-i}}{4}, && \text{if } n \text{ is even} \\ \sigma_{-n} &= 2\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} (-1)^{i+1} 8^{\frac{n-1}{2}-i}, && \text{if } n \text{ is odd.} \end{aligned}$$

*Proof.* By using (3-7), the *nth* power of $P$ is

$$P^n = \begin{bmatrix} x_n - 2x_{n-1} & -2x_{n-1} \\ -2x_{n-1} & x_n - 6x_{n-1} \end{bmatrix}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left( \begin{array}{c} n-i \\ i \end{array} \right) 8^{n-2i}(-8)^i = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left( \begin{array}{c} n-i \\ i \end{array} \right) 8^{n-i}(-1)^i$. We get the result by the equality of corresponding entries of (1,2)-th. Firstly, if $n$ is an even number,

$$-2x_{n-1} = -2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left( \begin{array}{c} n-1-i \\ i \end{array} \right) 8^{n-1-i}(-1)^i = 8^{\frac{n}{2}} \rho_{-n},$$

and if $n$ is an odd number

$$-2x_{n-1} = -2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left( \begin{array}{c} n-1-i \\ i \end{array} \right) 8^{n-1-i}(-1)^i = 8^{\frac{n-1}{2}} \sigma_{-n}.$$

□

Note that, for $n = 4$ we can write $\rho_{-4} = \sum_{i=0}^{2} \binom{3-i}{i} (-1)^{i+1} \frac{8^{2-i}}{4} = -16 + 4 - 0 = -12$.

**Theorem 2.3.** *For $n$, $k$ positive numbers, we have*

$$\begin{aligned} \rho_{-nk} &= \rho_{-n} \sigma_{-n}^{k-1} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \left( \begin{array}{c} k-1-i \\ i \end{array} \right) (-1)^{i(n+1)} \sigma_{-n}^{-2i}, \quad \textit{if } n \quad \textit{is even,} \\ \sigma_{-nk} &= \rho_{-n}^{k-1} \sigma_{-n} 8^{k-1} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \left( \begin{array}{c} k-1-i \\ i \end{array} \right) (-1)^{ni} 8^{-i} \rho_{-n}^{-2i}, \quad \textit{if } n, \ k \textit{ are odd}. \end{aligned}$$

*Proof.* If $n$ is an even number, the $kth$ power of the matrix $P^n$ is demonstrated by (5) as

$$P^{nk} = \left[ \begin{array}{cc} x_k - 8^{\frac{n}{2}} \rho_{-(n-1)} x_{k-1} & 8^{\frac{n}{2}} \rho_{-n} x_{k-1} \\ 8^{\frac{n}{2}} \rho_{-n} x_{k-1} & x_k - 8^{\frac{n}{2}} \rho_{-(n+1)} x_{k-1} \end{array} \right]$$

where $x_k = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \left( \begin{array}{c} k-i \\ i \end{array} \right) (8^{\frac{n}{2}} \sigma_{-n})^{k-2i} (-(-8)^n)^i$, since by (3), (6) as
$\det(P^n) = 8^n \left( \rho_{-(n+1)} \rho_{-(n-1)} - \rho_{-n}^2 \right) = (-8)^n$ and $tr(P^n) = 8^{\frac{n}{2}} (\rho_{-(n+1)} + \rho_{-(n-1)}) = 8^{\frac{n}{2}} \sigma_{-n}$. If we substitute for $n \to nk$, it gives us the $nk.th$ power of $P$ as the following

$$P^{nk} = 8^{\frac{nk}{2}} \left[ \begin{array}{cc} \rho_{-(nk+1)} & \rho_{-nk} \\ \rho_{-nk} & \rho_{-(nk-1)} \end{array} \right].$$

By the equality of corresponding entries of the matrices, the desired result is obtained as

$$\begin{aligned} 8^{\frac{n}{2}} \rho_{-n} x_{k-1} &= 8^{\frac{n}{2}} \rho_{-n} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \left( \begin{array}{c} k-1-i \\ i \end{array} \right) (8^{\frac{n}{2}} \sigma_{-n})^{k-1-2i} (-(-8)^n)^i \\ &= 8^{\frac{nk}{2}} \rho_{-nk}. \end{aligned}$$

If $n, k$ are odd numbers, we have

$$P^{nk} = \left[ \begin{array}{cc} x_k - 8^{\frac{n-1}{2}} \sigma_{-(n-1)} x_{k-1} & 8^{\frac{n-1}{2}} \sigma_{-n} x_{k-1} \\ 8^{\frac{n-1}{2}} \sigma_{-n} x_{k-1} & x_k - 8^{\frac{n-1}{2}} \sigma_{-(n+1)} x_{k-1} \end{array} \right]$$

where $x_k = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \left( \begin{array}{c} k-i \\ i \end{array} \right) (8^{\frac{n+1}{2}} \rho_{-n})^{k-2i} ((-1)^n 8^n)^i$ because of (5), (7) , $\det(P^n) = 8^{n-1} \left( \sigma_{-(n+1)} \sigma_{-(n-1)} - \sigma_{-n}^2 \right) = (-1)^{n-1} 8^n$ and $tr(P^n) = 8^{\frac{n-1}{2}} (\sigma_{-(n+1)} + \sigma_{-(n-1)}) = 8^{\frac{n+1}{2}} \rho_{-n}$. The result is obtained by the equality of the matrices as

$$\begin{aligned} 8^{\frac{n-1}{2}} \sigma_{-n} x_{k-1} &= 8^{\frac{n-1}{2}} \sigma_{-n} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \left( \begin{array}{c} k-1-i \\ i \end{array} \right) (8^{\frac{n+1}{2}} \rho_{-n})^{k-1-2i} (-8)^{ni} \\ &= 8^{\frac{nk-1}{2}} \sigma_{-nk}. \end{aligned}$$

□

Note that, for $n = 2, k = 3$ we get
$\rho_{-6} = \rho_{-2} \sigma_{-2}^2 \sum_{i=0}^{1} \left( \begin{array}{c} 2-i \\ i \end{array} \right) (-1)^{3i} \sigma_{-2}^{-2i} = -2(36)(1 - \frac{1}{36}) = -70.$

**Corollary 2.4.** *By matrix product, the following identities are established*

$$\begin{aligned} \rho_{-(n+1)} \rho_{-m} + \rho_{-n} \rho_{-(m-1)} &= \rho_{-(n+m)}, \quad \text{if} \quad m, n \quad \text{even}, \\ \sigma_{-(n+1)} \sigma_{-m} + \sigma_{-n} \sigma_{-(m-1)} &= 8 \rho_{-(n+m)}, \quad \text{if} \quad m, n \quad \text{odd}, \\ \rho_{-(n+1)} \sigma_{-m} + \rho_{-n} \sigma_{-(m-1)} &= \sigma_{-(n+m)}, \quad \text{if} \quad n \quad \text{even}, \quad m \quad \text{odd}, \\ \rho_{-(m+1)} \sigma_{-n} + \rho_{-m} \sigma_{-(n-1)} &= \sigma_{-(n+m)}, \quad \text{if} \quad m \quad \text{even}, \quad n \quad \text{odd}. \end{aligned}$$

*Proof.* If $m$, $n$ are even numbers, then $m+n$ is also even number. By Theorem 2.3 it is satisfied that

$$\begin{aligned} P^m P^n &= 8^{\frac{m+n}{2}} \left[ \begin{array}{cc} \rho_{-(m+1)} & \rho_{-m} \\ \rho_{-m} & \rho_{-(m-1)} \end{array} \right] \left[ \begin{array}{cc} \rho_{-(n+1)} & \rho_{-n} \\ \rho_{-n} & \rho_{-(n-1)} \end{array} \right] \\ &= P^{m+n} = 8^{\frac{m+n}{2}} \left[ \begin{array}{cc} \rho_{-(m+n+1)} & \rho_{-(m+n)} \\ \rho_{-(m+n)} & \rho_{-(m+n-1)} \end{array} \right]. \end{aligned}$$

By the equality of (2,1)-th elements of matrices, the result is obtained. The other results are also found by similar way. □

**Corollary 2.5.** *By matrix product, we get*

$$\begin{aligned} \sigma_{-(m+1)} \rho_{-n} + \sigma_{-m} \rho_{-(n+1)} &= \sigma_{-(m-n)}, \quad \text{or} \\ -\rho_{-m} \rho_{-(n-1)} + \rho_{-(m-1)} \rho_{-n} &= \rho_{-(m-n)}, \quad \text{if} \quad m, n \quad \text{even}, \end{aligned}$$

$$\begin{aligned} -\sigma_{-m} \sigma_{-(n-1)} + \sigma_{-(m-1)} \sigma_{-n} &= 8 \rho_{-(m-n)}, \quad \text{if} \quad m, n \quad \text{odd}, \\ -\rho_{-m} \sigma_{-(n-1)} + \rho_{-(m-1)} \sigma_{-n} &= \sigma_{-(m-n)}, \quad \text{if} \quad m \quad \text{even}, \quad n \quad \text{odd}, \\ -\sigma_{-m} \rho_{-(n-1)} + \sigma_{-(m-1)} \rho_{-n} &= \sigma_{-(m-n)}, \quad \text{if} \quad m \quad \text{odd}, \quad n \quad \text{even}. \end{aligned}$$

*Proof.* If $m$, $n$ are even numbers, then $m + n$ is also even number. By Theorem 2.3, it is satisfied:

$$\begin{aligned} P^m P^{-n} &= 8^{\frac{m}{2}} \left[ \begin{array}{cc} \rho_{-(m+1)} & \rho_{-m} \\ \rho_{-m} & \rho_{-(m-1)} \end{array} \right] . 8^{\frac{n}{2}} \left[ \begin{array}{cc} \rho_{-(n-1)} & -\rho_{-n} \\ -\rho_{-n} & \rho_{-(n+1)} \end{array} \right] \left( \frac{1}{(-8)^n} \right) \\ &= P^{m-n} = 8^{\frac{m-n}{2}} \left[ \begin{array}{cc} \rho_{-(m-n+1)} & \rho_{-(m-n)} \\ \rho_{-(m-n)} & \rho_{-(m-n-1)} \end{array} \right]. \end{aligned}$$

By the equality of (2,1)-th elements of matrices, the result is obtained. The other results are also found similarly. □

Note that, for $n=2$, $m=3$ we get

$$\begin{aligned} -\sigma_{-(m+1)}\rho_{-n}+\sigma_{-m}\rho_{-(n+1)} &= \sigma_{-(m-n)} \\ -\sigma_{-4}\rho_{-2}+\sigma_{-3}\rho_{-3} &= \sigma_{-1} \\ -34(-2)+5(-14) &= -2 \end{aligned}$$

**Theorem 2.6.** *Let us assume that $n$, $r$, $k$ are positive integers, the following identities are satisfied:*

*For $k$, $n$, $r$ even,*

$$\rho_{-(nk+r)}=\sigma_{-n}^{k}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k-i}{i}\sigma_{-n}^{-2i}(-1)^{i}\left[\rho_{-r}-\frac{k-2i}{k-i}\frac{\rho_{-(r-n)}}{\sigma_{-n}}\right].$$

*For $k,n,r$ odd,*

$$\rho_{-(nk+r)}=8^{\frac{k-1}{2}}\rho_{-n}^{k}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k-i}{i}\rho_{-n}^{-2i}(-8)^{-i}\left[\sigma_{-r}+\frac{k-2i}{k-i}\frac{\rho_{-(r-n)}}{\rho_{-n}}\right].$$

*For $n$ odd, $k$, $r$ even,*

$$\rho_{-(nk+r)}=8^{\frac{k}{2}}\rho_{-n}^{k}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k-i}{i}\rho_{-n}^{-2i}(-8)^{-i}\left[\rho_{-r}+\frac{k-2i}{k-i}\frac{\sigma_{-(n-r)}}{8\rho_{-n}}\right].$$

*For $n$, $r$ even, $k$ odd,*

$$\rho_{-(nk+r)}=\sigma_{-n}^{k}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k-i}{i}\sigma_{-n}^{-2i}(-1)^{i}\left[\rho_{-r}-\frac{k-2i}{k-i}\frac{\rho_{-(r-n)}}{\sigma_{-n}}\right].$$

*For $n$, $r$ odd, $k$ even,*

$$\sigma_{-(nk+r)}=8^{\frac{k}{2}}\rho_{-n}^{k}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k-i}{i}\rho_{-n}^{-2i}(-8)^{-i}\left[\sigma_{-r}+\frac{k-2i}{k-i}\frac{\rho_{-(n-r)}}{\rho_{-n}}\right].$$

*For $k$, $r$ odd, $n$ even,*

$$\sigma_{-(nk+r)}=\sigma_{-n}^{k}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k-i}{i}\sigma_{-n}^{-2i}(-1)^{i}\left[\sigma_{-r}+\frac{k-2i}{k-i}\frac{\sigma_{-(n-r)}}{\sigma_{-n}}\right].$$

*For $k$, $n$ odd, $r$ even,*

$$\sigma_{-(nk+r)}=\rho_{-n}^{k}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k-i}{i}\rho_{-n}^{-2i}(-8)^{-i}\left[\rho_{-r}-\frac{k-2i}{k-i}\frac{\sigma_{-(n-r)}}{8\rho_{-n}}\right].$$

*For $k$, $n$ even, $r$ odd,*

$$\sigma_{-(nk+r)}=\sigma_{-n}^{k}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k-i}{i}\sigma_{-n}^{-2i}(-1)^{i}\left[\sigma_{-r}+\frac{k-2i}{k-i}\frac{\sigma_{-(n-r)}}{\sigma_{-n}}\right].$$

*Proof.* Assume that $k$, $n$, $r$ even integers, then $nk+r$ is even. By using Theorem 2.1 , it is obtained that

$$P^{nk+r}=8^{\frac{nk+r}{2}}\begin{bmatrix}\rho_{-(nk+r+1)} & \rho_{-(nk+r)}\\ \rho_{-(nk+r)} & \rho_{-(nk+r-1)}\end{bmatrix}.$$

Then by (5), it is obtained that

$$\begin{aligned}P^{nk+r} &= (P^n)^k P^r = 8^{\frac{r}{2}}\begin{bmatrix} y_k - 8^{\frac{n}{2}}\rho_{-(n-1)}y_{k-1} & 8^{\frac{n}{2}}\rho_{-n}y_{k-1}\\ 8^{\frac{n}{2}}\rho_{-n}y_{k-1} & y_k - 8^{\frac{n}{2}}\rho_{-(n+1)}y_{k-1}\end{bmatrix}\\ &\quad .\begin{bmatrix}\rho_{-(r+1)} & \rho_{-r}\\ \rho_{-r} & \rho_{-(r-1)}\end{bmatrix}\end{aligned}$$

where $y_k=\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\begin{pmatrix}k-i\\ i\end{pmatrix}(8^{\frac{n}{2}}\sigma_{-n})^{k-2i}(-(-8)^n)^i$. By the equality of matrices, it is obtained that

$$8^{\frac{r}{2}}[\left(y_k-8^{\frac{n}{2}}\rho_{-(n-1)}y_{k-1}\right)\rho_{-r}+8^{\frac{n}{2}}\rho_{-n}y_{k-1}\rho_{-(r-1)}]=8^{\frac{nk+r}{2}}\rho_{-(nk+r)}.$$

After some algebraic operation, we get

$$\begin{aligned}8^{\frac{nk}{2}}\rho_{-(nk+r)} &= \rho_{-r}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\begin{pmatrix}k-i\\ i\end{pmatrix}(8^{\frac{n}{2}}\sigma_{-n})^{k-2i}(-(-8)^n)^i\\ &\quad -8^{\frac{n}{2}}\rho_{-r}\rho_{-(n-1)}\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\begin{pmatrix}k-1-i\\ i\end{pmatrix}(8^{\frac{n}{2}}\sigma_{-n})^{k-1-2i}(-(-8)^n)^i\\ &\quad +8^{\frac{n}{2}}\rho_{-n}\rho_{-(r-1)}\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\begin{pmatrix}k-1-i\\ i\end{pmatrix}(8^{\frac{n}{2}}\sigma_{-n})^{k-1-2i}(-(-8)^n)^i.\end{aligned}$$

$$\begin{aligned}&= 8^{\frac{nk}{2}}\rho_{-r}\sigma_{-n}^k\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\begin{pmatrix}k-i\\ i\end{pmatrix}\sigma_{-n}^{-2i}(-1)^{ni+i}\\ &\quad +8^{\frac{nk}{2}}\sigma_{-n}^{k-1}\rho_{-(r-n)}\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\begin{pmatrix}k-1-i\\ i\end{pmatrix}\sigma_{-n}^{-2i}(-1)^{ni+i+n-1}\\ &= 8^{\frac{nk}{2}}\sigma_{-n}^k\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\begin{pmatrix}k-i\\ i\end{pmatrix}\sigma_{-n}^{-2i}(-1)^i\left[\rho_{-r}-\frac{k-2i}{k-i}\frac{\rho_{-(r-n)}}{\sigma_{-n}}\right].\end{aligned}$$

$$\begin{aligned}&= \rho_{-r}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\begin{pmatrix}k-i\\ i\end{pmatrix}(8^{\frac{n}{2}}\sigma_{-n})^{k-2i}(-(-8)^n)^i\\ &\quad +\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\begin{pmatrix}k-1-i\\ i\end{pmatrix}\begin{pmatrix}(8^{\frac{n}{2}}\sigma_{-n})^{k-1-2i}(-(-8)^n)^i 8^{\frac{n}{2}}\\ .(-\rho_{-r}\rho_{-(n-1)}+\rho_{-n}\rho_{-(r-1)})\end{pmatrix}.\end{aligned}$$

For the other proofs, a similar way is used. □

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