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Integer Nonlinear Programming Formulation for some Variations of Paired Domination in Multigraphs without Loops

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Abstract

In this paper, we construct integer programming formulations for the paired, twin paired, paired restrained, and outer paired dominating set problems.

Let G be a multigraph. A set $S \subseteq V(G)$ is called a *paired dominating set* (resp., an *outer paired dominating set*) if it is a dominating set in G and $\langle S \rangle$ (resp., $\langle S^c \rangle$) contains at least one perfect matching. The *paired domination number* $\gamma_p(G)$ (resp., *outer paired domination number* $\gamma_{op}(G)$) is defined to be the minimum cardinality of a paired (resp., an outer paired) dominating set S in G. Moreover, a set $S \subseteq V(G)$ is called a *twin paired dominating set* (resp., *paired restrained dominating set*) in G if S is a paired dominating set and $\langle S^c \rangle$ contains a perfect matching (resp., contains no isolated vertex). The *twin paired domination number* $\gamma_{tp}(G)$ (resp., *paired restrained domination number* $\gamma_{pr}(G)$) is defined to be the minimum cardinality of a twin paired (resp., paired restrained) dominating set S in G.

1 Introduction and Preliminaries

Mathematical optimization is a branch of applied mathematics which is useful in different fields like engineering, mechanics, networks, manufacturing, transportation, finance, etc. Optimization comes from the same root as "optimal" which means best. But "best" can vary. If you're running a business, you would want to maximize your profit and minimize the cost. Both maximizing and minimizing are types of optimization problems.

Graph theory is a field of mathematics that many researchers have taken interest of, because of its diverse applications such as solving puzzles, describing physical network system, its use to engineering, computer science, and many more.

The notion of domination is one of the fundamental concept of graph theory that is extensively studied by many mathematicians. The concept of domination has historical roots as early as 1850s, when European enthusiast studied the problem "dominating queens" as described in [17]. A century later, mathematical study of dominating sets began in earnest, and since then, dominating sets have been used for different applications.

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We will discuss some concepts mentioned in [1], [5], [6], [7], [8], [10] [13], and [17] that is needed in this study. A *graph* $G = (V(G), E(G))$ is a pair consisting of a finite nonempty set $V(G)$ of objects called *vertices* together with a (possible empty) set $E(G)$ of unordered pairs of distinct vertices of G called *edges*. The set $V(G)$ and $E(G)$ are called *vertex set* and *edge set*, respectively. The *order* of a graph G is the number of vertices in G and is denoted by $|V(G)|$, and its *size* is the number of edges in G and is denoted by $|E(G)|$. A *multigraph* is a triple $(V(G), E(G), f_G)$ consisting of two disjoint sets $V(G)$ and $E(G)$, and a map $f_G : E(G) \longrightarrow V(G) \cup [V(G)]^2$, where $[V(G)]^2 := \{\{u, v\} : u, v \in V(G), u \neq v\}$, assigning to every edge either one or two vertices, its ends. From now onward, we let G be a multigraph. Let v_i and v_j be vertices in G. If v_i and v_j are joined by an edge e in G, then v_i and v_j are said to be *adjacent*. Moreover, v_i and v_j are said to be *incident* with e, and e is said to be *incident* with v_i and v_j. In this case, we write $e \sim v_iv_j$ where $v_i, v_j \in V(G)$. Two edges e_1 and e_2 of G are *adjacent edges* if e_1 and e_2 are incident to a common vertex in G. Two edges in G are *independent* if they are not adjacent in G. If two or more edges join in the same pair of distinct vertices, then these edges are called *parallel edges* or *multiple edges*. If an edge e joins a vertex v to itself, then e is called a *loop*. Observe that a graph is thus essentially the same as a multigraph without loops or multiple edges. The number of edges incident to a vertex v_i of G is called the *degree* of v_i and is denoted by $\deg_G(v_i)$. If $\deg_G(v_i) = 0$, then v_i is called an *isolated vertex*. The smallest and the largest degree among the vertices of G is denoted by $\delta(G)$ or simply δ, and $\Delta(G)$ or simply Δ, respectively. For vertex v of G, the *open neighborhood* $N_G(v)$ of the vertex v consists of the set of vertices adjacent to v, that is, $N_G(v) = \{w \in V(G) : vw \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighborhood $N_G(S)$ is defined to be $\cup_{v \in S} N_G(v)$, and the closed neighborhood of S is $N_G[S] = N_G(S) \cup S$. The multigraph whose vertex set is S and whose edge set comprises exactly the edges in G which join vertices in S is termed an *induced subgraph* of G and is denoted by $\langle S \rangle$. A set of pairwise independent edges in G is called a *matching* in G. If M is a matching in G with the property that every vertex of G is incident with an edge of M, then M is a *perfect matching* in G. A matching M is said to be *maximal* if M is not properly contained in any other matching. Formally, $M \nsubseteq M'$ for any matching M' of G with $M' \neq M$. Intuitively, this is equivalent to saying that a matching is maximal if we cannot add any edge to the existing set such that the resulting set is still a matching. A matching M is said to be *maximum* if for any other matching M', $|M| \geq |M'|$. $|M|$ is the maximum sized matching.

In 1998, Haynes and Slater [16] introduced a domination parameter called paired domination. In 2015, Wang *et al.* [2] introduced another domination parameter called outer paired domination. A set $S \subseteq V(G)$ is called a *paired dominating set* (resp., an *outer paired dominating set*) if it is a dominating set in G and $\langle S \rangle$ (resp., $\langle S^c \rangle$) contains at least one perfect matching. The *paired domination number* $\gamma_p(G)$ (resp., *outer paired domination number* $\gamma_{op}(G)$) is defined to be the minimum cardinality of a paired (resp., an outer paired) dominating set S in G. If a paired (resp., an outer paired) dominating set S has $|S| = \gamma_p(G)$ (resp., $|S| = \gamma_{op}(G)$), we say that S is a *minimum paired dominating set* (resp., *minimum outer paired dominating set*). Haynes and Slater suggests that the paired domination can be used to model situation in which the dominating set is a set of guards and each guard is assigned to another adjacent guard and they are designated as backups for each other. Also, the outer paired domination can be used to model situation in which each nodes $v \in V(G)$ are classified into two roles: job-tracker and task-tracker, where each task-tracker is monitored by a job-tracker. And when a task-tracker failed to do its task, the job-tracker then transfers the task to its backup task tracker.

In 2020, Mahadevan and Suganthi [9] defined a new variant of paired domination called twin paired domination. The following year, Hung and Chiu [14] introduced another variant of paired domination called the paired restrained domination. A set $S \subseteq V(G)$ is called a *twin* *paired dominating set* (resp., *paired restrained dominating set*) in G if S is a paired dominating set and $\langle S^c \rangle$ contains a perfect matching (resp., contains no isolated vertex). The *twin paired domination number* $\gamma_{tp}(G)$ (resp., *paired restrained domination number* $\gamma_{pr}(G)$) is defined to be the minimum cardinality of a twin paired (resp., paired restrained) dominating set S in G. If a twin paired (resp., paired restrained) dominating set S has $|S| = \gamma_{tp}(G)$ (resp. $|S| = \gamma_{pr}(G)$), we say that S is a *minimum twin paired dominating set* (resp., *minimum paired restrained dominating set*). The paired restrained domination has a possible application in the system of prisoners and guards. Each nodes in $V(G)$ represents the position of a guard or prisoner. To maintain safety and support, each guard's location is observed by exactly another guard's location, and each prisoner's location is monitored by at least a guard's location to maintain security, and each prisoner's location is seen by another prisoner's location to protect the rights of prisoners.

2 INLP Formulation

2.1 Paired Dominating Set (PDS) Problem

We will now present an INLP formulation for the PDS problem.
For a multigraph G of order $n \geq 2$, without loops and isolated vertices, let $V(G) := \{v_1, v_2, \ldots, v_n\}$ and $E(G) := \{e_1, e_2, \ldots, e_m\}$. Let $S \subseteq V(G)$ and $M \subseteq E(G)$, and for all $v_i \in V(G)$, let $x_i \in \{0, 1\}$ be a decision variable defined by

$$x_i := \begin{cases} 1, & \text{if } v_i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Also, for all $e_k \in E(G)$, let $y_k \in \{0, 1\}$ be another decision variable defined by

$$y_k := \begin{cases} 1, & \text{if } e_k \in M, \\ 0, & \text{otherwise.} \end{cases}$$

An INLP formulation for the PDS problem can be given as follows:

$$\min \sum_{i=1}^{n} x_i \tag{P1}$$

subject to

$$A\mathbf{x} \geq \vec{1}_n \tag{P2}$$

$$\sum_{\substack{(j,k):\\ v_i v_j \sim e_k \in E(G)}} x_i x_j y_k = x_i \qquad \forall i \in \{1, 2, \ldots, n\} \tag{P3}$$

$$\mathbf{x} \in \{0, 1\}^n \tag{P4}$$

$$\mathbf{y} \in \{0, 1\}^m \tag{P5}$$

where $A := [a_{ij}]$ is an $n \times n$ matrix defined by

$$a_{ij} := \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, and $\vec{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ with n rows.

Theorem 2.1.

(*i*) *If S is a paired dominating set in G and M is a perfect matching in $\langle S\rangle$, then $(\mathbf{x}, \mathbf{y})$ is a solution to the INLP formulation for the PDS problem.*

(*ii*) *If $(\mathbf{x}, \mathbf{y})$ is a solution to the INLP formulation for the PDS problem, then S is a paired dominating set in G and M contains a perfect matching in $\langle S\rangle$.*

Moreover, the optimal solution value of the INLP formulation for the PDS problem is equal to the paired domination number of G. Furthermore, M is a perfect matching in $\langle S\rangle$ if and only if $\sum_{k=1}^{m} y_k$ is minimized for $\mathbf{y} \in \{0,1\}^m$ in a solution $(\mathbf{x}, \mathbf{y})$.

Proof. (i) Let S be a paired dominating set in G and let M be a perfect matching in $\langle S\rangle$. Let $i \in \{1, 2, \ldots, n\}$ with its corresponding vertex $v_i \in V(G)$, and let $k \in \{1, 2, \ldots, m\}$ with its corresponding edge $e_k \in E(G)$. Note that constraints (P4) and (P5) are trivially satisfied by the definition of decision variables x_i's and y_k's. Let $v_i \in v(G)$. Consider the following cases.
Case 1. $v_i \in V(G) \setminus S$.
Then $x_i = 0$. Since S is a paired-dominating set in G, there exists $v_j \in S$ such that v_i and v_j are adjacent in G. Thus, $x_j = 1 = a_{ij}$ so that $a_{ij}x_j = 1$. And so, $\sum_{j=1}^{n} a_{ij}x_j \geq 1$. Also,

$$\sum_{\substack{(j,k):\\ v_iv_j \sim e_k \in E(G)}} x_ix_jy_k = 0 = x_i.$$

Thus, constraints (P2) and (P3) are satisfied.
Case 2. $v_i \in S$.
Since M is a perfect matching in $\langle S\rangle$, there exists $v_j \in S$ and $e_k \in M$ such that $e_k \sim v_iv_j$. Thus, $x_j = 1 = a_{ij}$ so that $a_{ij}x_j = 1$. And so, $\sum_{j=1}^{n} a_{ij}x_j \geq 1$. Hence, constraint (P2) is satisfied. Next, we show that constraint (P3) is also satisfied. Since $e_k \in M$ and $v_i, v_j \in S$, $x_i = x_j = y_k = 1$. And so $x_ix_jy_k = 1$. We will show that $x_ix_jy_k$ is the only term in constraint (P3) in which $x_ix_jy_k = 1$. Let $(j^*, k^*) \neq (j, k)$ with $v_iv_{j^*} \sim e_{k^*}$. Consider the following subcases.
Subcase 1. $j^* \neq j$.
Then e_{k^*} is adjacent to e_k. Since $e_k \in M$, $e_{k^*} \notin M$ so that $y_k^* = 0$. Hence, $x_ix_{j^*}y_{k^*} = 0$.
Subcase 2. $j^* = j$ and $k^* \neq k$.
Then $e_{k^*} \sim v_iv_j$, $e_{k^*} \neq e_k$, and e_{k^*} is adjacent to e_k. Since $e_k \in M$, $e_{k^*} \notin M$ so that $y_k^* = 0$. Hence, $x_ix_jy_{k^*} = 0$. And so,

$$\sum_{\substack{(j,k):\\ v_iv_j \sim e_k \in E(G)}} x_ix_jy_k = x_ix_jy_k = 1 = x_i.$$

Thus, in either case, constraints (P2) and (P3) are satisfied.

Since constraints (P2), (P3), (P4), and (P5) are satisfied, the pair $(\mathbf{x}, \mathbf{y})$ is a solution to the INLP formulation for the PDS problem.

(ii) Suppose that the pair $(\mathbf{x}, \mathbf{y})$ is a solution to the INLP formulation for the PDS problem. Then, (P2), (P3), (P4), and (P5) are satisfied. Let $S := \{v_i \in V(G) : x_i = 1\}$ and $M = \{e_k \in E(G) : y_k = 1\}$. Let $v_i \in V(G)$. By constraint (P2),

$$A\mathbf{x} \geq \vec{1},$$

In particular, corresponding to the i^{th} row, we have

$$\sum_{j=1}^{n} a_{ij}x_j \geq 1.$$

Hence, there exists $j \in \{1, 2, \ldots, n\}$ such that $a_{ij}x_j = 1$ so that $a_{ij} = 1$ and $x_j = 1$. Thus, v_j and v_i are adjacent in G where $v_j \in S$, so that S is a dominating set in G. In fact, S is a total dominating set.

Since we have shown that for $v_i, v_j \in S$, $v_iv_j \in E(G)$, we have $|S| \geq 2$ and $\langle S \rangle$ contains a matching. Let $v_i \in S$. Then $x_i = 1$. By constraint (P3),

$$\sum_{\substack{(j,k)\,: \\ v_iv_j \sim e_k \in E(G)}} x_jy_k = \sum_{\substack{(j,k)\,: \\ v_iv_j \sim e_k \in E(G)}} x_ix_jy_k = x_i = 1.$$

Hence, there exists a unique $(j, k) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ such that $e_k \sim v_iv_j \in E(G)$ and $x_jy_k = 1$. Thus, $x_j = y_k = 1$ so that $v_j \in S$ and $e_k \in M$. This guarantees that there exists $e_k \sim v_iv_j \in M$ with $v_i, v_j \in S$. Define $M' = \{e_k \in M : e_k \sim v_iv_j \text{ for some } v_i, v_j \in S\}$. We are left to show that M' is a perfect matching in $\langle S \rangle$ so that S is a paired dominating set in G.
Let $e_{k_1} \in M'$ so that $e_{k_1} \sim v_{i_1}v_{j_1}$ for some $v_{i_1}, v_{j_1} \in S$. Then, $x_{i_1} = x_{j_1} = y_{k_1} = 1$. To show that M' is a matching, suppose, on the contrary, that there exists $e_{k'} \in M'$, $e_{k'} \neq e_{k_1}$ such that $e_{k'}$ is adjacent to e_{k_1}. Then $y_{k'} = 1$. Without loss of generality, assume that v_{i_1} is incident to $e_{k'}$.
Case 1. Suppose that $e_{k'} \sim v_{i_1}v_l$ with $v_l \in S$, $v_l \neq v_{j_1}$. Then, $x_l = y_{k'} = 1$ so that $x_ly_{k'} = 1$. Hence, $x_{i_1}x_{j_1}y_{k_1} + x_{i_1}x_ly_{k'} = 1 + 1 = 2$, a contradiction to constraint (P3).
Case 2. Suppose that $e_{k'} \sim v_{i_1}v_{j_1}$. Then, $x_{i_1} = x_{j_1} = y_{k'} = 1$. Hence, $x_{i_1}x_{j_1}y_{k_1} + x_{i_1}x_{j_1}y_{k'} = 1 + 1 = 2$, a contradiction to constraint (P3).
Hence, M' is a matching in $\langle S \rangle$. M' is also a perfect matching in $\langle S \rangle$ since we have already shown from above that for each $v_i \in S$, there exists $e_k \in M'$ such that $e_k \sim v_iv_j$ for some $v_j \in S$. And so, S is a paired dominating set in G and $M' \subseteq M$ is a perfect matching in $\langle S \rangle$.

For the next statement of the theorem, let S be a minimum paired dominating set in G and M be a perfect matching in $\langle S \rangle$. Then $\gamma_p(G) = |S|$ and the pair $(\mathbf{x}, \mathbf{y})$ is a feasible solution to the objective function (P1). Thus,

$$\sum_{i=1}^{n} x_i = |S| = \gamma_p(G)$$

so that

$$\min \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} x_i = \gamma_p(G).$$

Conversely, let $S = \{v_i \in V(G) : x_i = 1\}$ such that $|S|$ is an optimal solution value to the objective function (P1). Then S is a paired doiminating set in G. Since the paired domination number of G is the minimum cardinality of paired dominating set in G, we have

$$\gamma_p(G) \leq |S| = \min \sum_{i=1}^{n} x_i.$$

Accordingly, the optimal solution value of the INLP formulation for the PDS problem is equal to the paired domination number of G.

Finally, for the final statement of the theorem, suppose that M is a perfect matching in $\langle S \rangle$, S is a paired dominating set in G. Then $|S| = 2|M|$. By (i), $(\mathbf{x}, \mathbf{y})$ is a solution to the INLP formulation for the PDS problem. Suppose that for $\mathbf{y} \in \{0,1\}^m$, $\sum_{k=1}^{m} y_k$ is not minimized. Then there exists $\mathbf{y}^* = (y_1^*, \ldots, y_m^*) \in \{0,1\}^m$ such that $(\mathbf{x}, \mathbf{y}^*)$ is a solution to the INLP formulation for the PDS problem, and $\sum_{k=1}^{m} y_k^* < \sum_{k=1}^{m} y_k$. By (ii), the set $M^* = \{e_k \in E(G) : y_k^* = 1\}$ contains a perfect matching $M' = \{e_k \in M^* : e_k \sim v_i v_j \text{ for some } v_i, v_j \in S\}$ in $\langle S \rangle$. Thus,

$$|M'| \leq |M^*| = \sum_{k=1}^{m} y_k^* < \sum_{k=1}^{m} y_k = |M|.$$

Since M' is perfect matching in $\langle S \rangle$, $|S| = 2|M'|$. Hence, we now have

$$|S| = 2|M'| < 2|M| = |S|,$$

a contradiction. Thus, for $\mathbf{y} \in \{0,1\}^m$, $\sum_{k=1}^{m} y_k$ is minimized.

Conversely, let $(\mathbf{x}, \mathbf{y})$ be a solution to the INLP formulation for the PDS problem. Assume that for $\mathbf{y} \in \{0,1\}^m$, $\sum_{k=1}^{m} y_k$ is minimized. Then $\sum_{k=1}^{m} y_k \leq \sum_{k=1}^{m} y_k^*$ for all $\mathbf{y}^* = (y_1^*, \ldots, y_m^*) \in \{0,1\}^m$ where $(\mathbf{x}, \mathbf{y}^*)$ is a solution to the INLP formulation for the PDS problem. By (ii), $S = \{v_i \in V(G) : x_i = 1\}$ is a paired dominating set in G and $M = \{e_k \in E(G) : y_k = 1\}$ contains a perfect matching $M' = \{e_k \in M : e_k \sim v_i v_j \text{ for some } v_i, v_j \in S\}$ in $\langle S \rangle$. Hence, by (i), $(\mathbf{x}, \mathbf{y}')$ is a solution to the INLP formulation for the PDS problem, where $\mathbf{y}' = (y_1', \ldots, y_m') \in \{0,1\}^m$ is the corresponding vector of M'. Thus,

$$|M| = \sum_{k=1}^{m} y_k \leq \sum_{k=1}^{m} y_k' = |M'|.$$

Since $M' \subseteq M$, $|M'| \leq |M|$. Hence, we have $|M| = |M'|$ so that $M = M'$. Therefore, M is a perfect matching in $\langle S \rangle$. □

2.2 Twin Paired Dominating Set (TPDS) Problem

Theorem 2.2. *Let G be a multigraph. Then G contains a twin paired dominating set $S \neq V(G)$ if and only if G has a perfect matching M and there exists $e \sim uv \in M$ such that $|N_G(u)|, |N_G(v)| \geq 2$.*

Proof. Assume that G contains a twin paired dominating set $S \neq V(G)$. Then S is a dominating set such that $\langle S \rangle$ contains a perfect matching M' and $\langle S^c \rangle$ contains a perfect matching M''. Let $M := M' \cup M''$. Since $S \cap S^c = \varnothing$, $M' \cap M'' = \varnothing$. Hence, M is a set of independent edges with vertex set $S \cup S^c = V(G)$. Thus, M is a perfect matching in G. Let $e \in M'' \subset M$. Then, $e \sim uv$ for some $u, v \in S^c$. Also, since S is a dominating set, there exist $u', v' \in S$ such that u' and u are adjacent in G, and v' and v are adjacent in G. Thus, $v, u' \in N_G(u)$ and $u, v' \in N_G(v)$ so that $|N_G(u)|, |N_G(v)| \geq 2$.

Conversely, suppose that G has a perfect matching M and there exists $e \sim uv \in M$ such that $|N_G(u)|, |N_G(v)| \geq 2$. Let $S := V(G) \setminus \{u, v\}$. Then $S \neq V(G)$.

Claim 1: S is a dominating set in G and $S \neq \varnothing$.

Let $a \in \{u, v\}$. Without loss of generality, assume that $a = u$. Since $|N_G(u)| \geq 2$, there exists $v' \in V(G)$, $v' \neq v$, such that v' and u are adjacent in G. Also, $v' \neq u$ since we do not allow loops here. Thus, $v' \notin \{u, v\}$ so that $v' \in S$. Hence, S is a dominating set in G and $S \neq \varnothing$.

Claim 2: The set $M \setminus \{e\}$ is a perfect matching in $\langle S \rangle$.

Since $M \setminus \{e\} \subset M$ and M is a perfect matching in G, it follows that $M \setminus \{e\}$ is a set of independent edges so that $M \setminus \{e\}$ is a matching in G. Now, since M is a perfect matching in G, for all $v^* \in V(G)$, there exist unique $u^* \in V(G)$ and $e_{k^*} \in M$ such that $e_{k^*} \sim u^*v^*$. Let $v' \in S$. Then, there exist unique $u' \in V(G)$ and $e_{k'} \in M$ such that $e_{k'} \sim u'v'$. Since $e \sim uv$ and $v' \notin \{u, v\}$, this implies that $e_{k'} \neq e$ so that $e_{k'} \in M \setminus \{e\}$. Furthermore, since M is a perfect matching in G, $e_{k'}$ and e are independent edges so that $u' \notin \{u, v\}$. Thus, $e_{k'} \sim u'v' \in M \setminus \{e\}$ with $u', v' \in S$. And so, $M \setminus \{e\}$ is a perfect matching in $\langle S \rangle$.

Also, since $e \sim uv$, it follows that $\{e\}$ is a perfect matching in $\langle \{u, v\} \rangle = \langle S^c \rangle$. Therefore, S is a twin paired dominating set in G. $\square$

We will now present an INLP formulation for the TPDS problem.

For a multigraph G of order $n \geq 2$, without loops and isolated vertices, let $V(G) := \{v_1, v_2, \ldots, v_n\}$ and $E(G) := \{e_1, e_2, \ldots, e_m\}$. Let $S \subseteq V(G)$, $M \subseteq E(G)$, $M' \subseteq E(G)$, and for all $v_i \in V(G)$, let $x_i \in \{0, 1\}$ be a decision variable defined by

$$x_i := \begin{cases} 1, & \text{if } v_i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Also, for all $e_k \in E(G)$, let $y_k \in \{0, 1\}$ be another decision variable defined by

$$y_k := \begin{cases} 1, & \text{if } e_k \in M, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for all $e_k \in E(G)$, let $z_k \in \{0, 1\}$ be another decision variable defined by

$$z_k := \begin{cases} 1, & \text{if } e_k \in M', \\ 0, & \text{otherwise.} \end{cases}$$

An INLP formulation for the TPDS problem can be given as follows:

$$\min \sum_{i=1}^{n} x_i \tag{TP1}$$

subject to

$$A\mathbf{x} \geq \vec{1}_n \tag{TP2}$$

$$\sum_{\substack{(j,k): \\ v_i v_j \sim e_k \in E(G)}} x_i x_j y_k = x_i \qquad \forall i \in \{1, 2, \ldots, n\} \tag{TP3}$$

$$\sum_{\substack{(j,k): \\ v_i v_j \sim e_k \in E(G)}} (1 - x_i)(1 - x_j) z_k = 1 - x_i \qquad \forall i \in \{1, 2, \ldots, n\} \tag{TP4}$$

$$\mathbf{x} \in \{0, 1\}^n \tag{TP5}$$

$$\mathbf{y} \in \{0, 1\}^m \tag{TP6}$$

$$\mathbf{z} \in \{0, 1\}^m \tag{TP7}$$

where $A := [a_{ij}]$ is an $n \times n$ matrix defined by

$$a_{ij} := \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$, and $\vec{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ with n rows.

Theorem 2.3.

(*i*) *If S is a twin paired dominating set in G, M is a perfect matching in $\langle S\rangle$, and M' is a perfect matching in $\langle S^c\rangle$, then $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a solution to the INLP formulation for the TPDS problem.*

(*ii*) *If $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a solution to the INLP formulation for the TPDS problem, then S is a twin paired dominating set in G, M contains a perfect matching in $\langle S\rangle$, and M' contains a perfect matching in $\langle S^c\rangle$.*

Moreover, the optimal solution value of the INLP formulation for the TPDS problem is equal to the twin paired domination number of G. Furthermore, M is a perfect matching in $\langle S\rangle$ and M' is a perfect matching in $\langle S^c\rangle$ if and only if $\sum_{k=1}^{m} y_k$ and $\sum_{k=1}^{m} z_k$ are minimized for $\mathbf{y}, \mathbf{z} \in \{0,1\}^m$ in a solution $(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

Proof. (*i*) Let S be a twin paired dominating set in G, M be a perfect matching in $\langle S\rangle$, and M' be a perfect matching in $\langle S^c\rangle$. Let $i \in \{1, 2, \ldots, n\}$ with its corresponding vertex $v_i \in V(G)$, and let $k \in \{1, 2, \ldots, m\}$ with its corresponding edge $e_k \in E(G)$. The proof is similar to theorem 2.1 to show that constraints (TP2), (TP3), (TP5), and (TP6) are satisfied. Note that constraint (TP7) is trivially satisfied by the definition of decision variable z_k's. Hence, we are left to show that constraint (TP4) must be satisfied. Let $v_i \in V(G)$. Consider the following cases.
Case 1:$v_i \in S$.
Then $x_i = 1$. Hence,

$$\sum_{\substack{(j,k):\\ v_iv_j \sim e_k \in E(G)}} (1 - x_i)(1 - x_j)z_k = 0 = 1 - x_i.$$

Thus, constraint (TP4) is satisfied.
Case 2: $v_i \notin S$.
Since M' is a perfect matching in $\langle S^c\rangle$, there exists $v_j \in S^c$ and $e_k \in M'$ such that $e_k \sim v_iv_j$. Thus, $x_i = x_j = 0$ and $z_k = 1$. And so $(1-x_i)(1-x_j)z_k = 1$. We will show that $(1-x_i)(1-x_j)z_k$ is the only term in constraint (TP4) in which $(1 - x_i)(1 - x_j)z_k = 1$. Let $(j^*, k^*) \neq (j, k)$ with $v_iv_{j^*} \sim e_{k^*}$. Consider the following subcases.
Subcase 1: $j^* \neq j$.
Then e_{k^*} is adjacent to e_k. Since $e_k \in M'$, $e_{k^*} \notin M'$ so that $z_{k^*} = 0$. Hence, $(1-x_i)(1-x_{j^*})z_{k^*} = 0$.
Subcase 2: $j^* = j$ and $k^* \neq k$.
Then $e_{k^*} \sim v_iv_j$, $e_{k^*} \neq e_k$, and e_{k^*} is adjacent to e_k. Since $e_k \in M'$, $e_{k^*} \notin M'$ so that $z^*_k = 0$. Hence, $(1 - x_i)(1 - x_j)z_{k^*} = 0$. And so,

$$\sum_{\substack{(j,k):\\ v_iv_j \sim e_k \in E(G)}} (1 - x_i)(1 - x_j)z_k = (1 - x_i)(1 - x_j)z_k = 1 = 1 - x_i.$$

Thus, in either case, (TP4) is satisfied.

Since constraints (TP2), (TP3), (TP4), (TP5), (TP6), and (TP7) are satisfied, the triple $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a solution to the INLP formulation for the TPDS problem.

(ii) Suppose that the triple $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a solution to the INLP formulation for the TPDS problem. Then, (TP2), (TP3), (TP4), (TP5), (TP6), and (TP7) are satisfied. Let $S := \{v_i \in V(G) : x_i = 1\}$, $M = \{e_k \in E(G) : y_k = 1\}$, and $M' = \{e_k \in E(G) : z_k = 1\}$. The proof is similar to theorem 2.1 for showing that S is a paired-dominating set in G and M contains a perfect matching in $\langle S \rangle$. Let $v_i \in S^c$. Then $x_i = 0$. By constraint (TP4),

$$\sum_{\substack{(j,k):\\ v_iv_j \sim e_k \in E(G)}} (1 - x_j)z_k = 1$$

Hence, there exists unique $(j, k) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ such that $e_k \sim v_iv_j \in E(G)$ and $(1-x_j)z_k = 1$. Thus, $x_j = 0$ and $z_k = 1$ so that $v_j \in S^c$ and $e_k \in M'$. This guarantees that there exists $e_k \sim v_iv_j \in M'$ with $v_i, v_j \in S^c$. Define $M^* = \{e_k \in M' : e_k \sim v_iv_j \text{ for some } v_i, v_j \in S^c\}$. We are left to show that M^* is a perfect matching in $\langle S^c \rangle$ so that S is a twin paired dominating set in G.
Let $e_{k_1} \in M^*$ so that $e_{k_1} \sim v_{i_1}v_{j_1}$ for some $v_{i_1}, v_{j_1} \in S^c$. Then, $x_{i_1} = x_{j_1} = 0$ and $z_{k_1} = 1$. To show that M^* is a matching, suppose, on the contrary, that there exists $e_{k'} \in M^*$, $e_{k'} \neq e_{k_1}$ such that $e_{k'}$ is adjacent to e_{k_1}. Then, $z_{k'} = 1$. Without loss of generality, assume that v_{i_1} is incident to $e_{k'}$.
Case 1: Suppose that $e_{k'} = v_{i_1}v_l$ with $v_l \in S^c$, $v_l \neq v_{j_1}$. Then, $x_l = 0$ so that $(1 - x_l)z_{k'} = 1$. Hence, $(1 - x_{i_1})(1 - x_{j_1})z_{k_1} + (1 - x_{i_1})(1 - x_l)z_{k'} = 1 + 1 = 2$, a contradiction to constraint (TP4).
Case 2: Suppose that $e_{k'} \sim v_{i_1}v_{j_1}$. Then, $x_{i_1} = x_{j_1} = 0$. Hence, $(1 - x_{i_1})(1 - x_{j_1})z_{k_1} + (1 - x_{i_1})(1 - x_{j_1})z_{k'} = 1 + 1 = 2$, a contradiction to constraint (TP4).
Hence, M^* is a matching in $\langle S^c \rangle$. M^* is also a perfect matching in $\langle S^c \rangle$ since we have already shown from above that for each $v_i \in S^c$, there exists $e_k \in M^*$ such that $e_k \sim v_iv_j$ for some $v_j \in S^c$. And so, S is a twin paired dominating set in G, M contains a perfect matching in $\langle S \rangle$, and $M^* \subseteq M'$ is a perfect matching in $\langle S^c \rangle$.

For the next statement of the theorem, let S be a minimum twin paired dominating set in G, M be a perfect matching in $\langle S \rangle$, and M' be a perfect matching in $\langle S^c \rangle$. Then $\gamma_{tp}(G) = |S|$ and the triple $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a feasible solution to the objective function (TP1). Thus,

$$\sum_{i=1}^{n} x_i = |S| = \gamma_{tp}(G)$$

so that

$$\min \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} x_i = \gamma_{tp}(G).$$

Conversely, let $S = \{v_i \in V(G) : x_i = 1\}$ such that $|S|$ is an optimal solution value to the objective function (TP1). Then S is a twin paired doiminating set in G. Since the twin paired domination number of G is the minimum cardinality of twin paired dominating set in G, we have

$$\gamma_{tp}(G) \leq |S| = \min \sum_{i=1}^{n} x_i.$$

Accordingly, the optimal solution value of the INLP formulation for the TPDS problem is equal to the twin paired domination number of G.

Finally, for the last statement of the theorem, suppose that M is a perfect matching in $\langle S \rangle$ and M' is a perfect matching in $\langle S^c \rangle$, where S is a twin paired dominating set in G. Then

$|S| = 2|M|$ and $|S^c| = 2|M'|$. By (i), $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a solution to the INLP formulation of the TPDS problem. Suppose that for $\mathbf{y}, \mathbf{z} \in \{0,1\}^m$, $\sum_{k=1}^{m} y_k$ is not minimized or $\sum_{k=1}^{m} z_k$ is not minimized.

Case 1: $\sum_{k=1}^{m} y_k$ is not minimized.
Then there exists $\mathbf{y}^* = (y_1^*, \ldots, y_m^*) \in \{0,1\}^m$ such that $(\mathbf{x}, \mathbf{y}^*, \mathbf{z})$ is a solution to the INLP formulation for the TPDS problem, and $\sum_{k=1}^{m} y_k^* < \sum_{k=1}^{m} y_k$. By (ii), the set $M^{1*} = \{e_k \in E(G) : y_k^* = 1\}$ contains a perfect matching $M^1 = \{e_k \in M^{1*} : e_k \sim v_i v_j \text{ for some } v_i, v_j \in S\}$ in $\langle S \rangle$. Thus,

$$|M^1| \leq |M^{1*}| = \sum_{k=1}^{m} y_k^* < \sum_{k=1}^{m} y_k = |M|.$$

Since M^1 is perfect matching in $\langle S \rangle$, $|S| = 2|M^1|$. Hence, we now have

$$|S| = 2|M^1| < 2|M| = |S|,$$

a contradiction. Thus, for $\mathbf{y} \in \{0,1\}^m$, $\sum_{k=1}^{m} y_k$ is minimized.

Case 2: $\sum_{k=1}^{m} z_k$ is not minimized.
Then there exists $\mathbf{z}^* = (z_1^*, \ldots, z_m^*) \in \{0,1\}^m$ such that $(\mathbf{x}, \mathbf{y}, \mathbf{z}^*)$ is a solution to the INLP formulation for the TPDS problem, and $\sum_{k=1}^{m} z_k^* < \sum_{k=1}^{m} z_k$. By (ii), the set $M^{2*} = \{e_k \in E(G) : z_k^* = 1\}$ contains a perfect matching $M^2 = \{e_k \in M^{2*} : e_k \sim v_i v_j \text{ for some } v_i, v_j \in S^c\}$ in $\langle S^c \rangle$. Thus,

$$|M^2| \leq |M^{2*}| = \sum_{k=1}^{m} z_k^* < \sum_{k=1}^{m} z_k = |M'|.$$

Since M^2 is perfect matching in $\langle S^c \rangle$, $|S^c| = 2|M^2|$. Hence, we now have

$$|S^c| = 2|M^2| < 2|M'| = |S^c|,$$

a contradiction. Thus, for $\mathbf{z} \in \{0,1\}^m$, $\sum_{k=1}^{m} z_k$ is minimized.

Therefore, for $\mathbf{y}, \mathbf{z} \in \{0,1\}^m$, $\sum_{k=1}^{m} y_k$ and $\sum_{k=1}^{m} z_k$ are minimized.

Conversely, let $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ be a solution to the INLP formulation for the TPDS problem. Assume that for $\mathbf{y}, \mathbf{z} \in \{0,1\}^m$, $\sum_{k=1}^{m} y_k$ and $\sum_{k=1}^{m} z_k$ are minimized. Then $\sum_{k=1}^{m} y_k \leq \sum_{k=1}^{m} y_k^*$ and $\sum_{k=1}^{m} z_k \leq \sum_{k=1}^{m} z_k^*$ for all $\mathbf{y}^*, \mathbf{z}^* \in \{0,1\}^m$, $\mathbf{y}^* = (y_1^*, \ldots, y_m^*)$ and $\mathbf{z}^* = (z_1^*, \ldots, z_m^*)$, where $(\mathbf{x}, \mathbf{y}^*, \mathbf{z}^*)$ is a solution to the INLP formulation for the TPDS problem. By (ii), $S = \{v_i \in V(G) : x_i = 1\}$ is a twin paired dominating set in G, $M = \{e_k \in E(G) : y_k = 1\}$ contains a perfect matching $M^1 = \{e_k \in M^1 : e_k \sim v_i v_j \text{ for some } v_i, v_j \in S\}$ in $\langle S \rangle$, and $M' = \{e_k \in E(G) : z_k = 1\}$ contains a perfect matching $M^2 = \{e_k \in M^2 : e_k \sim v_i v_j \text{ for some } v_i, v_j \in S^c\}$ in $\langle S^c \rangle$. Hence, by (i), $(\mathbf{x}, \mathbf{y}', \mathbf{z}')$ is a solution to the INLP formulation of the TPDS problem, where $\mathbf{y}' = (y_1', \ldots, y_m') \in \{0,1\}^m$ and $\mathbf{z}' = (z_1', \ldots, z_m') \in \{0,1\}^m$ are the corresponding vectors of M^1 and M^2 respectively. Thus,

$$|M| = \sum_{k=1}^{m} y_k \leq \sum_{k=1}^{m} y_k' = |M^1|$$

and

$$|M'| = \sum_{k=1}^{m} z_k \leq \sum_{k=1}^{m} z'_k = |M^2|.$$

Since $M^1 \subseteq M$ and $M^2 \subseteq M'$, $|M^1| \leq |M|$ and $|M^2| \leq |M'|$. Hence, we have $|M| = |M^1|$ and $|M'| = |M^2|$ so that $M = M^1$ and $M' = M^2$. Therefore, M is a perfect matching in $\langle S \rangle$ and M' is a perfect matching in $\langle S^c \rangle$. □

2.3 Paired Restrained Dominating Set (PRDS) Problem

For the next result, we denote by S_M the set of vertices from $M \subseteq E(G)$.

Theorem 2.4. *Let G be a multigraph. Then G contains a paired restrained dominating set $S \neq V(G)$ if and only if G contains a matching M such that for all $w \notin S_M$, $0 \neq |N_G(w) \cap S_M| < |N_G(w)|$.*

Proof. Assume that G contains a paired restrained dominating set $S \neq V(G)$. Then S is a dominating set, $\langle S \rangle$ contains a perfect matching M, and for each $a \notin S$, there exists $b \notin S$ such that a and b are adjacent in G. Thus, G contains a matching M and S is the set of vertices from M so that $S = S_M$. Let $w \notin S_M$. Since S_M is a dominating set, there exists $u \in S_M$ such that u and w are adjacent in G. Thus, $u \in N_G(w) \cap S_M$ so that $|N_G(w) \cap S_M| \neq 0$. Note that $N_G(w) \cap S_M \subseteq N_G(w)$. Since S_M is a paired restrained-dominating set, and $w \notin S_m$, there exists $v \notin S_M$ such that v and w are adjacent in G. Thus, $v \in N_G(w)$ and $v \notin N_G(w) \cap S_M$ so that $N_G(w) \cap S_M \subset N_G(w)$. Therefore, $0 \neq |N_G(w) \cap S_M| < |N_G(w)|$.

Conversely, suppose that G contains a matching M such that for all $w \notin S_M$, $0 \neq |N_G(w) \cap S_M| < |N_G(w)|$. Thus, $S_M \neq V(G)$ since $|N_G(w) \cap S_M| < |N_G(w)|$ for all $w \notin S_M$. Let $S := S_M$. Then $S \neq V(G)$ and M is a perfect matching in $\langle S \rangle$. Let $w' \in V(G) \setminus S$. By assumption, $0 \neq |N_G(w') \cap S| < |N_G(w')|$. Hence, there exist $u \in S$ and $v \notin S$ such that u and w' are adjacent in G, and v and w' are also adjacent in G. Therefore, S is a dominating set and $\langle S^c \rangle$ contains no isolated vertex. Since M is a perfect matching in $\langle S \rangle$, it follows that S is a paired restrained dominating set in G. □

We will now present an INLP formulation for the PRDS problem.
For a multigraph G of order $n \geq 2$, without loops and isolated vertices, let $V(G) := \{v_1, v_2, \dots, v_n\}$ and $E(G) := \{e_1, e_2, \dots, e_m\}$. Let $S \subseteq V(G)$ and $M \subseteq E(G)$, and for all $v_i \in V(G)$, let $x_i \in \{0, 1\}$ be a decision variable defined by

$$x_i := \begin{cases} 1, & \text{if } v_i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Also, for all $e_k \in E(G)$, let $y_k \in \{0, 1\}$ be another decision variable defined by

$$y_k := \begin{cases} 1, & \text{if } e_k \in M, \\ 0, & \text{otherwise.} \end{cases}$$

An INLP formulation for the PRDS problem can be given as follows:

$$\min \sum_{i=1}^{n} x_i \tag{PR1}$$

subject to

$$A\mathbf{x} \geq \vec{1}_n \tag{PR2}$$

$$\sum_{\substack{(j,k):\\ v_iv_j \sim e_k \in E(G)}} x_i x_j y_k = x_i \qquad \forall i \in \{1,2,\ldots,n\} \tag{PR3}$$

$$\sum_{j=1}^{n} b_{ij}x_j < |N_G(v_i)| \qquad \forall i \in \{1,2,\ldots,n\} \tag{PR4}$$

$$\mathbf{x} \in \{0,1\}^n \tag{PR5}$$

$$\mathbf{y} \in \{0,1\}^m \tag{PR6}$$

where $A := [a_{ij}]$ is an $n \times n$ matrix defined by

$$a_{ij} := \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$b_{ij} := \begin{cases} -1, & \text{if } v_i = v_j, \\ 1, & v_i \text{ and } v_j \text{ are adjacent in } G, \\ 0, & \text{otherwise .} \end{cases}$$

$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, and $\vec{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ with n rows.

Theorem 2.5.

(*i*) *If S is a paired restrained dominating set in G and M is a perfect matching in $\langle S \rangle$, then $(\mathbf{x}, \mathbf{y})$ is a solution to the INLP formulation for the PRDS problem.*

(*ii*) *If $(\mathbf{x}, \mathbf{y})$ is a solution to the INLP formulation for the PRDS problem, then S is a paired restrained dominating set in G and M contains a perfect matching in $\langle S \rangle$.*

Moreover, the optimal solution value of the INLP formulation for the PRDS problem is equal to the paired restrained domination number of G. Furthermore, M is a perfect matching in $\langle S \rangle$ if and only if $\sum_{k=1}^{m} y_k$ is minimized for $\mathbf{y} \in \{0,1\}^m$ in a solution $(\mathbf{x}, \mathbf{y})$.

Proof. (*i*) Let S be a paired restrained dominating set in G and M be a perfect matching in $\langle S \rangle$. Let $i \in \{1,2,\ldots,n\}$ with its corresponding vertex $v_i \in V(G)$, and let $k \in \{1,2,\ldots,m\}$ with its corresponding edge $e_k \in E(G)$.
The proof is similar to theorem 2.1 to show that constraints (PR2), (PR3), (PR5), and (PR6) are satisfied. Hence, we are left to show that constraint (PR4) must be satisfied. Let $v_i \in V(G)$. Consider the following cases.
Case 1: $v_i \in S$
Then $x_i = 1$. Since S is a paired restrained dominating set in G, for all $v_j \notin S$ such that v_j and v_i are adjacent in G, $b_{ij} = 1$ and $x_j = 0$ so that $b_{ij}x_j = 0$.
Also, since S is a paired restrained dominating set in G, $\langle S \rangle$ contains no isolated vertex. Thus, for all $v_j \in S$ such that v_j and v_i are adjacent in G, $b_{ij} = 1$ and $x_j = 1$ so that $b_{ij}x_j = 1$. Hence,

$$\sum_{j=1, j\neq i}^{n} b_{ij}x_j = |N_G(v_i) \cap S|.$$

Since $b_{ii} = -1$ and $x_i = 1$, $b_{ii}x_i = -1$. Thus,

$$\sum_{j=1}^{n} b_{ij}x_j = b_{ii}x_i + \sum_{j=1, j\neq i}^{n} c_{ij}x_j = -1 + |N_G(v_i) \cap S| < |N_G(v_i)|.$$

Thus, constraint (PR4) is satisfied.
Case 2: $v_i \notin S$.
Then $x_i = 0$. Since S is a paired restrained-dominating set in G, $\langle S^c \rangle$ contains no isolated vertex. Thus, for all $v_j \notin S$ such that v_j and v_i are adjacent in G, $b_{ij} = 1$ and $x_j = 0$ so that $b_{ij}x_j = 0$.
Also, since S is a paired restrained-dominating set, $\langle S \rangle$ contains no isolated vertex. Thus, for all $v_j \in S$ such that v_j and v_i are adjacent in G, $b_{ij} = 1$ and $x_j = 1$ so that $b_{ij}x_j = 1$. Hence,

$$\sum_{j=1, j\neq i}^{n} b_{ij}x_j = |N_G(v_i) \cap S|.$$

Since $b_{ii} = -1$ and $x_i = 0$, $b_{ii}x_i = 0$. Thus,

$$\sum_{j=1}^{n} b_{ij}x_j = |N_G(v_i) \cap S| < |N_G(v_i)|.$$

Thus, in either cases, constraint (PR4) is satisfied.

Since constraints (PR2), (PR3), (PR4), (PR5),and (PR6) are satisfied, the pair $(\mathbf{x}, \mathbf{y})$ is a solution to the INLP formulation for the PRDS problem.

(ii) Suppose that the pair $(\mathbf{x}, \mathbf{y})$ is a solution to the INLP formulation for the PRDS problem. Then, (PR2), (PR3), (PR4), (PR5),and (PR6) are satisfied. Let $S := \{v_i \in V(G) : x_i = 1\}$ and $M = \{e_k \in E(G) : y_k = 1\}$. The proof is similar to theorem 2.1 for showing that S is a paired dominating set in G. We will show that $\langle S^c \rangle$ contains no isolated vertex. Let $v_i \in S^c$. Then, $x_i = 0$ so that $b_{ii}x_i = 0$. By constraint (PR4),

$$\sum_{j=1}^{n} b_{ij}x_j < |N_G(v_i)|.$$

Observe that

$$\sum_{j=1}^{n} b_{ij}x_j = |N_G(v_i) \cap S|$$

so that

$$|N_G(v_i) \cap S| < |N_G(v_i)|.$$

This implies that there exists $v_j \in S^c$ such that v_j and v_i are adjacent in G. Hence, $\langle S^c \rangle$ contains no isolated vertex. And so, S is a paired restrained dominating set of G.

For the next statement of the theorem, let S be a minimum paired restrained dominating set in G and M be a perfect matching in $\langle S \rangle$. Then $\gamma_{pr}(G) = |S|$ and the pair $(\mathbf{x}, \mathbf{y})$ is a feasible solution to the objective function (PR1). Thus,

$$\sum_{i=1}^{n} x_i = |S| = \gamma_{pr}(G)$$

so that

$$\min \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} x_i = \gamma_{pr}(G).$$

Conversely, let $S = \{v_i \in V(G) : x_i = 1\}$ such that $|S|$ is an optimal solution value to the objective function (PR1). Then S is a paired restrained doiminating set in G. Since the paired restrained domination number of G is the minimum cardinality of paired restrained dominating set in G, we have

$$\gamma_{pr}(G) \leq |S| = \min \sum_{i=1}^{n} x_i.$$

Accordingly, the optimal solution value of the INLP formulation for the PRDS problem is equal to the paired restrained domination number of G.

For the last statement of the theorem, the proof is similar to theorem 2.1 in showing that M is a perfect matching in $\langle S \rangle$ if and only if $\mathbf{y} \in \{0,1\}^m$ such that $\sum_{k=1}^{m} y_k$ is minimized. □

2.4 Outer Paired Dominating Set (OPDS) Problem

Theorem 2.6. *Let G be a multigraph. Then G contains an outer paired dominating set $S \neq V(G)$ if and only if there exists $u, v \in V(G)$ such that u and v are adjacent in G and $|N_G(u)|, |N_G(v)| \geq 2$.*

Proof. Assume that G contains an outer paired dominating set $S \neq V(G)$. Then S is a dominating set in G and $\langle S^c \rangle$ contains a perfect matching M. Let $e \in M$. Then, there exist $u, v \in S^c \subset V(G)$ such that $e \sim uv$, so that u and v are adjacent in G. Also, since S is a dominating set, there exist $u', v' \in S$ such that u' and u are adjacent in G, and v' and v are adjacent in G. Thus, $v, u' \in N_G(u)$ and $u, v' \in N_G(v)$ so that $|N_G(u)|, |N_G(v)| \geq 2$.

Conversely, suppose that there exists $u, v \in V(G)$ such that u and v are adjacent in G and $|N_G(u)|, |N_G(v)| \geq 2$. Let $S := V(G) \setminus \{u, v\}$. Then $S \neq V(G)$.

Claim: S is a dominating set in G and $S \neq \varnothing$.

Let $a \in \{u, v\}$. Without loss of generality, assume that $a = u$. Since $|N_G(u)| \geq 2$, there exists $v' \in V(G)$, $v' \neq v$, such that v' and u are adjacent in G. Also, $v' \neq u$ since we do not allow loops here. Thus, $v' \notin \{u, v\}$ so that $v' \in S$. Hence, S is a dominating set in G and $S \neq \varnothing$.

Now, since u and v are adjacent in G, there exists $e \in E(G)$ such that $e \sim uv$. Thus, $\{e\}$ is a perfect matching in $\langle \{u, v\} \rangle = \langle S^c \rangle$. Therefore, S is an outer paired dominating set in G. □

We will now present an INLP formulation for the OPDS problem.

For a multigraph G of order $n \geq 3$, without loops and isolated vertices, let $V(G) := \{v_1, v_2, \ldots, v_n\}$ and $E(G) := \{e_1, e_2, \ldots, e_m\}$. Let $S \subseteq V(G)$ and $M' \subseteq E(G)$, and for all $v_i \in V(G)$, let $x_i \in \{0, 1\}$ be a decision variable defined by

$$x_i := \begin{cases} 1, & \text{if } v_i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Also, for all $e_k \in E(G)$, let $z_k \in \{0, 1\}$ be another decision variable defined by

$$z_k := \begin{cases} 1, & \text{if } e_k \in M', \\ 0, & \text{otherwise.} \end{cases}$$

An INLP formulation for the OPDS problem can be given as follows:

$$\min \sum_{i=1}^{n} x_i \tag{OP1}$$

subject to

$$A\mathbf{x} \geq \vec{1}_n \tag{OP2}$$

$$\sum_{\substack{(j,k):\\ v_iv_j \sim e_k \in E(G)}} (1-x_i)(1-x_j)z_k = 1 - x_i \qquad \forall i \in \{1,2,\ldots,n\} \tag{OP3}$$

$$\mathbf{x} \in \{0,1\}^n \tag{OP4}$$

$$\mathbf{z} \in \{0,1\}^m \tag{OP5}$$

where $A := [a_{ij}]$ is an $n \times n$ matrix defined by

$$a_{ij} := \begin{cases} 1, & \text{if } v_i = v_j \text{ or } v_i \text{ and } v_j \text{ are adjacent in } G, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$, and $\vec{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ with n rows.

Theorem 2.7.

(*i*) *If S is an outer paired dominating set in G and M' is a perfect matching in $\langle S^c \rangle$, then $(\mathbf{x}, \mathbf{z})$ is a solution to the INLP formulation for the OPDS problem.*

(*ii*) *If $(\mathbf{x}, \mathbf{z})$ is a solution to the INLP formulation for the OPDS problem, then S is an outer paired dominating set in G and M' contains a perfect matching in $\langle S^c \rangle$.*

Moreover, the optimal solution value of the INLP formulation for the OPDS problem is equal to the outer paired domination number of G. Furthermore, M' is a perfect matching in $\langle S^c \rangle$ if and only if $\sum_{k=1}^{m} z_k$ is minimized for $\mathbf{z} \in \{0,1\}^m$ in a solution $(\mathbf{x}, \mathbf{z})$.

Proof. (*i*) Let S be an outer paired dominating set in G and let M' be a perfect matching in $\langle S^c \rangle$. Let $i \in \{1, 2, \ldots, n\}$ with its corresponding vertex $v_i \in V(G)$, and let $k \in \{1, 2, \ldots, m\}$ with its corresponding edge $e_k \in E(G)$. Note that (OP4) and (OP5) are trivially satisfied by the definition of decision variables x_i's and z_k's. Let $v_i \in v(G)$. Consider the following cases.

Case 1. $v_i \in V(G) \setminus S$.

Then $x_i = 0$. Since S is an outer paired dominating set in G, there exists $v_h \in S$ such that v_i and v_h are adjacent in G. Thus, $x_h = 1 = a_{ih}$ so that $a_{ih}x_h = 1$. And so, $\sum_{j=1}^{n} a_{ij}x_j \geq 1$. Thus, constraint (OP2) is satisfied. Since M' is a perfect matching in $\langle S^c \rangle$, there exists $v_j \in S^c$ and $e_k \in M'$ such that $e_k \sim v_iv_j$. Thus, $x_i = x_j = 0$ and $z_k = 1$. And so, $(1-x_i)(1-x_j)z_k = 1$. We will show that $(1-x_i)(1-x_j)z_k$ is the only term in constraint (OP3) in which $(1-x_i)(1-x_j)z_k = 1$. Let $(j^*, k^*) \neq (j, k)$ with $v_iv_{j^*} \sim e_{k^*}$. Consider the following subcases.

Subcase 1: $j^* \neq j$.

Then e_{k^*} is adjacent to e_k. Since $e_k \in M'$, $e_{k^*} \notin M'$ so that $z_k^* = 0$. Hence, $(1-x_i)(1-x_{j^*})z_{k^*} = 0$.

Subcase 2: $j^* = j$ and $k^* \neq k$.
Then $e_{k^*} \sim v_i v_j$, $e_{k^*} \neq e_k$, and e_{k^*} is adjacent to e_k. Since $e_k \in M'$, $e_{k^*} \notin M'$ so that $z_k^* = 0$. Hence, $(1 - x_i)(1 - x_j)z_{k^*} = 0$. And so,

$$\sum_{\substack{(j,k):\\ v_iv_j \sim e_k \in E(G)}} (1 - x_i)(1 - x_j)z_k = (1 - x_i)(1 - x_j)z_k = 1 = 1 - x_i.$$

Thus, constraint (OP3) is satisfied.
Case 2. $v_i \in S$.
Then $x_i = 1$. Take $v_j = v_i$. Thus, $x_j = 1 = a_{ij}$ so that $a_{ij}x_j = 1$. And so, $\sum_{j=1}^{n} a_{ij}x_j \geq 1$. Also,

$$\sum_{\substack{(j,k):\\ v_iv_j \sim e_k \in E(G)}} (1 - x_i)(1 - x_j)z_k = 0 = 1 - x_i.$$

Thus, in either case, constraints (OP2) and (OP3) are satisfied.
Since constraints (OP2), (OP3), (OP4), and (OP5) are satisfied, the pair $(\mathbf{x}, \mathbf{z})$ is a solution to the INLP formulation for the OPDS problem.

(ii) Suppose that the pair $(\mathbf{x}, \mathbf{z})$ is a solution to the INLP formulation for the OPDS problem. Then, (OP2), (OP3), (OP4), and (OP5) are satisfied. Let $S := \{v_i \in V(G) : x_i = 1\}$ and $M' = \{e_k \in E(G) : z_k = 1\}$. Let $v_i \in V(G) \setminus S$. By constraint (OP2),

$$A\mathbf{x} \geq \vec{1},$$

In particular, corresponding to the i^{th} row, we have

$$\sum_{j=1}^{n} a_{ij}x_j \geq 1.$$

Hence, there exists $h \in \{1, 2, \ldots, n\}$ such that $a_{ih}x_h = 1$ so that $a_{ih} = 1$ and $x_h = 1$. Thus, $v_h \in S$ so that $v_h \neq v_i$. Furthermore, v_h and v_i are adjacent in G, so that S is a dominating set in G.

Since $v_i \in S^c$, $x_i = 0$. By constraint (OP3),

$$\sum_{\substack{(j,k):\\ v_iv_j \sim e_k \in E(G)}} (1 - x_j)z_k = 1$$

Hence, there exists unique $(j, k) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ such that $e_k \sim v_iv_j \in E(G)$ and $(1-x_j)z_k = 1$. Thus, $x_j = 0$ and $z_k = 1$ so that $v_j \in S^c$ and $e_k \in M'$. This guarantees that there exists $e_k \sim v_iv_j \in M'$ with $v_i, v_j \in S^c$. Define $M^* = \{e_k \in M' : e_k \sim v_iv_j \text{ for some } v_i, v_j \in S^c\}$. We are left to show that M^* is a perfect matching in $\langle S^c \rangle$ so that S is an outer paired dominating set in G.
Let $e_{k_1} \in M^*$ so that $e_{k_1} \sim v_{i_1}v_{j_1}$ for some $v_{i_1}, v_{j_1} \in S^c$. Then, $x_{i_1} = x_{j_1} = 0$ and $z_{k_1} = 1$. To show that M^* is a matching, suppose, on the contrary, that there exists $e_{k'} \in M^*$, $e_{k'} \neq e_{k_1}$ such that $e_{k'}$ is adjacent to e_{k_1}. Then, $z_{k'} = 1$. Without loss of generality, assume that v_{i_1} is incident to $e_{k'}$.
Case 1: Suppose that $e_{k'} = v_{i_1}v_l$ with $v_l \in S^c$, $v_l \neq v_{j_1}$. Then, $x_l = 0$ so that $(1 - x_l)z_{k'} = 1$. Hence, $(1 - x_{i_1})(1 - x_{j_1})z_{k_1} + (1 - x_{i_1})(1 - x_l)z_{k'} = 1 + 1 = 2$, a contradiction to constraint

(6.4.3).
Case 2: Suppose that $e_{k'} \sim v_{i_1}v_{j_1}$. Then, $x_{i_1} = x_{j_1} = 0$. Hence, $(1 - x_{i_1})(1 - x_{j_1})z_{k_1} + (1 - x_{i_1})(1 - x_{j_1})z_{k'} = 1 + 1 = 2$, a contradiction to constraint (6.4.3).
Hence, M^* is a matching in $\langle S^c \rangle$. M^* is also a perfect matching in $\langle S^c \rangle$ since we have already shown from above that for each $v_i \in S^c$, there exists $e_k \in M^*$ such that $e_k \sim v_iv_j$ for some $v_j \in S^c$. And so, S is an outer paired dominating set in G and $M^* \subseteq M'$ is a perfect matching in $\langle S^c \rangle$.

For the next statement of the theorem, let S be a minimum outer paired dominating set in G and M' be a perfect matching in $\langle S^c \rangle$. Then $\gamma_{op}(G) = |S|$ and the pair $(\mathbf{x}, \mathbf{z})$ is a feasible solution to the objective function (OP1). Thus,

$$\sum_{i=1}^{n} x_i = |S| = \gamma_{op}(G)$$

so that

$$\min \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} x_i = \gamma_{op}(G).$$

Conversely, let $S = \{v_i \in V(G) : x_i = 1\}$ such that $|S|$ is an optimal solution value to the objective function (OP1). Then S is an outer paired dominating set in G. Since the outer paired domination number of G is the minimum cardinality of outer paired dominating set in G, we have

$$\gamma_{op}(G) \leq |S| = \min \sum_{i=1}^{n} x_i.$$

Accordingly, the optimal solution value of the INLP formulation for the OPDS problem is equal to the outer paired domination number of G.

Finally, for the final statement of the theorem, suppose that M' is a perfect matching in $\langle S^c \rangle$, S is an outer-paired dominating set in G. Then $|S^c| = 2|M'|$. By (i), $(\mathbf{x}, \mathbf{z})$ is a solution to the INLP formulation for the OPDS problem. Suppose that for $\mathbf{z} \in \{0,1\}^m$, $\sum_{k=1}^{m} z_k$ is not minimized. Then there exists $\mathbf{z}^* = (z_1^*, \ldots, z_m^*) \in \{0,1\}^m$ such that $(\mathbf{x}, \mathbf{z}^*)$ is a solution to the INLP formulation for the OPDS problem, and $\sum_{k=1}^{m} z_k^* < \sum_{k=1}^{m} z_k$. By (ii), the set $M^* = \{e_k \in E(G) : z_k^* = 1\}$ contains a perfect matching $M^1 = \{e_k \in M^* : e_k \sim v_iv_j \text{ for some } v_i, v_j \in S^c\}$ in $\langle S^c \rangle$. Thus,

$$|M^1| \leq |M^*| = \sum_{k=1}^{m} z_k^* < \sum_{k=1}^{m} z_k = |M'|.$$

Since M^1 is perfect matching in $\langle S^c \rangle$, $|S^c| = 2|M^1|$. Hence, we now have

$$|S^c| = 2|M^1| < 2|M'| = |S^c|,$$

a contradiction. Thus, for $\mathbf{z} \in \{0,1\}^m$, $\sum_{k=1}^{m} z_k$ is minimized.

Conversely, let $(\mathbf{x}, \mathbf{z})$ be a solution to the INLP formulation for the OPDS problem. Assume that for $\mathbf{z} \in \{0,1\}^m$, $\sum_{k=1}^{m} z_k$ is minimized. Then $\sum_{k=1}^{m} z_k \leq \sum_{k=1}^{m} z_k^*$ for all $\mathbf{z}^* = (z_1^*, \ldots, z_m^*) \in \{0,1\}^m$ where $(\mathbf{x}, \mathbf{z}^*)$ is a solution to the INLP formulation for the OPDS problem. By (ii), $S = \{v_i \in$

$V(G) : x_i = 1\}$ is an outer paired dominating set in G and $M' = \{e_k \in E(G) : z_k = 1\}$ contains a perfect matching $M^1 = \{e_k \in M' : e_k \sim v_i v_j$ for some $v_i, v_j \in S^c\}$ in $\langle S^c \rangle$. Hence, by (i), $(\mathbf{x}, \mathbf{z}')$ is a solution to the INLP formulation of the OPDS problem, where $\mathbf{z}' = (z'_1, \ldots, z'_m) \in \{0, 1\}^m$ is the corresponding vector of M^1. Thus,

$$|M'| = \sum_{k=1}^{m} z_k \leq \sum_{k=1}^{m} z'_k = |M^1|.$$

Since $M^1 \subseteq M'$, $|M^1| \leq |M'|$. Hence, we have $|M'| = |M^1|$ so that $M' = M^1$. Therefore, M' is a perfect matching in $\langle S^c \rangle$. □

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