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On $\mathcal{F}$-Atoms and μ-Atoms of a Measure Space, and Atoms of Derived σ-Algebras

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Abstract

This study explores the concepts of $\mathcal{F}$-atoms and μ-atoms within a measure space and examines their interrelationship. Conditions are identified under which the atoms of a σ-algebra coincide with the atoms of a measure. The study also investigates $\mathcal{F}$-atoms in trace σ-algebra and pre-image σ-algebra. The concept of *quarks* is introduced as a generalization of $\mathcal{F}$-atoms which is then used in characterizing atoms in product σ-algebra.

1 Introduction

The notion of an atom in mathematics was first formally introduced and defined by G. Birkhoff, in his book "Lattice Theory" in 1940 [2]. It refers to a minimal nonzero element in a partially ordered set, particularly within a lattice. Later, in 1950, P. R. Halmos, in his book "Measure theory" [10], redefined the concept of an atom in the context of measure theory, describing atoms as indivisible elements in a measure ring. Since then, the term "atom" has been applied to many areas in mathematics, specifically atoms in a measure space and atoms of a σ-algebra. The notion of atoms of a measure as well as the purely atomic (or atomic) measure was introduced by E. Marczewski in 1955 [13]. He investigated when convergence in measure implies convergence almost everywhere or, respectively, convergence everywhere. In 1961, R. R. Phelps [15] characterized purely atomic measure spaces within the framework of Banach spaces. Nearly a decade later, in 1970, R. A. Johnson [11] further explored the properties of atomic and non-atomic measures. Moreover, his work demonstrated that every measure can be uniquely decomposed as the sum of a purely atomic measure and a non-atomic measure. More recently, in 2020, A. J. Al-Afloogee and N. F. Al-Mayahi [1] investigated normed spaces of measurable functions, discussing their properties in relation to atomic measures. In 2022, following the R. A. Johnson, A. Milazzo and P. Siorpaes [14] introduced an abstract decomposition of measures and analyzed how the outputs of the decomposition depend on its inputs. The concept of atoms

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has also been explored within σ-algebras. In 2008, J. Zhang [20] investigated conditions under which an atom of a σ-algebra can be determined from its generator. More recently, in 2022, S. Rao and A. Steinicke [16], examined the distributivity of products for σ-algebras with respect to intersection. Their findings showed that the existence of atoms significantly influences measure-theoretic operations, particularly in product spaces. In most of these studies, the authors use the concept of atoms in a σ-algebra and atoms in a measure as a condition to achieve their main objectives. In this study, we focus more directly on these concepts of atoms of a σ-algebra and atoms of a measure. We characterize atoms of a σ-algebra on some σ-algebras. Additionally, this paper investigates the relationship between atoms of a σ-algebra and atoms of a measure in a measure space.

2 Preliminaries

We will discuss some concepts mentioned in [6], [7], [8], [16], [18], and [20] that is needed in this study. In the following, we let Ω be a nonempty set. A collection $\mathcal{S} \subseteq 2^{\Omega}$ is called a *set system* if it contains the empty set, that is, $\emptyset \in \mathcal{S}$. A collection $\mathcal{D} \subseteq 2^{\Omega}$ is a *Dynkin system* or *δ-system* if the following conditions hold; (i) $\Omega \in \mathcal{D}$, (ii) If $D \in \mathcal{D}$, then $D^c \in \mathcal{D}$ (closed under complements), (iii) If $(D_j)_{j\in\mathbb{N}}$ is a sequence of pairwise disjoint sets such that $D_j \in \mathcal{D}$ for all j, then $\bigcup_{j\in\mathbb{N}} D_j \in \mathcal{D}$. We say that $\mathcal{P} \subseteq 2^{\Omega}$ is a *π-system* if it is closed under intersection, that is, if $E, F \in \mathcal{G}$ then $E \cap F \in \mathcal{P}$. A collection $\mathcal{G} \subseteq 2^{\Omega}$ is called an *algebra* on Ω if it is closed under finite intersection and complement, and $\Omega, \emptyset \in \mathcal{G}$. In addition, $\mathcal{F} \subseteq 2^{\Omega}$ is called a *σ-algebra* on Ω if it is closed under countable intersection and complement, and $\Omega, \emptyset \in \mathcal{F}$. Let $\mathcal{F}$ be a σ-algebra on Ω. The pair $(\Omega, \mathcal{F})$ is called *measurable space.*

For every collection $\mathcal{S} \subseteq 2^{\Omega}$, there exists a smallest (also called minimal or coarsest) σ-algebra containing $\mathcal{S}$. This is the *σ-algebra generated by* $\mathcal{S}$, denoted by $\sigma(\mathcal{S})$, and $\mathcal{S}$ is called its *generator.* A *topological space* $(\Omega, \mathcal{T})$ consists of a set $\Omega \neq \emptyset$ and a collection $\mathcal{T} \subseteq 2^{\Omega}$, called a *topology*, which satisfies the following properties; (i) $\emptyset, \Omega \in \mathcal{T}$, (ii) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$ and (iii) If $\{U_i\}_{i\in I}$ is an arbitrary collection of sets in $\mathcal{T}$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$. A set $U \in \mathcal{T}$ is called an *open set.* A set $F \subset \Omega$ is *closed* if its complement F^c is open. The σ-algebra $\sigma(\mathcal{O}^n)$ generated by the open sets $\mathcal{O}^n$ of $\mathbb{R}^n$ is called the *Borel σ-algebra*, and its members are the *Borel sets* or *Borel measurable sets.* We write $\mathcal{B}(\mathbb{R}^n)$ or $\mathcal{B}^n$ for the Borel sets in $\mathbb{R}^n$. Let $\mathcal{F}$ be a σ-algebra on Ω. A non-empty set $A \in \mathcal{F}$ is said to be an *atom* if and only if for all $F \in \mathcal{F}$, $F \subsetneq A$ implies $F = \emptyset$. A σ-algebra $\mathcal{F}$ is called *atomic* if Ω is the union of the atoms of $\mathcal{F}$. A collection $\mathcal{C} \subseteq 2^{\Omega}$ is called a *κ-class*, if it is closed under countable intersection and countable union. Denote by $\kappa(\mathcal{C})$ the minimum κ-class containing $\mathcal{C}$. The set $\mathcal{C}$ is called the *generator* of $\kappa(\mathcal{C})$. A *(positive) measure* on a set Ω is a mapping $\mu : \mathcal{F} \to [0, \infty]$ defined on a σ-algebra $\mathcal{F}$ satisfying; (M1) $\mu(\emptyset) = 0$. (M2) (σ-Additivity) For any countable family of pairwise disjoint sets $(F_j)_{j\in\mathbb{N}} \subseteq \mathcal{F}$, $\mu\left(\bigcup_{j\in\mathbb{N}} F_j\right) = \sum_{j\in\mathbb{N}} \mu(F_j)$. Let $\mathcal{F}$ be a σ-algebra on Ω, and μ a measure on Ω. The triple $(\Omega, \mathcal{F}, \mu)$ is called a *measure space.* A measure μ defined on $(\Omega, \mathcal{F})$ will be called *finite* if $\mu(\Omega) < \infty$. The measure μ is *σ-finite* if $\Omega = \bigcup_{i=1}^{\infty} F_i, \quad F_i \in \mathcal{F}, \quad \text{and} \quad \mu(F_i) < \infty$ for $i = 1, 2, \ldots$

Let $\mathcal{F}$ be a σ-algebra on Ω, and μ be a (nonnegative and σ-additive) σ-finite measure on $(\Omega, \mathcal{F})$. A set $M \in \mathcal{F}$ is an *atom* of the measure μ if $\mu(M) > 0$ and for every $B \in \mathcal{F}$, $B \subseteq M$, either $\mu(B) = 0$ or $\mu(B) = \mu(M)$. We say that a measure μ is (*purely*) *atomic* if every measurable set of strictly positive measure contains an atom of μ, and *diffuse* (or *non-atomic*) if $\mathcal{F}$ contains no atom of μ. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. We will call the atoms of $\mathcal{F}$ as *$\mathcal{F}$-atoms* and atoms of a measure μ as *μ-atoms.*

3 $\mathcal{F}$-Atoms and μ-Atoms

This section explores the relationship between $\mathcal{F}$-atoms and μ-atoms in a measure space. We first define the following; Let $(\Omega, \mathcal{F})$ be a measurable space, where $\mathcal{F}$ is a σ-algebra on Ω. For any $\omega \in \Omega$, define $\mathcal{F}_\omega = \{B \in \mathcal{F} \mid \omega \in B\}$ and $A_\omega = \bigcap_{B \in \mathcal{F}_\omega} B$.

Theorem 3.1. *Let* $(\Omega, \mathcal{F})$ *be a measurable space and* F *an* $\mathcal{F}$*-atom containing* ω*. Then* $F = A_\omega$.

Proof. Let F be an $\mathcal{F}$-atom and $\omega \in F$. Then $F \in \mathcal{F}$ and $\emptyset \neq A_\omega \subseteq F$. Since F is an $\mathcal{F}$-atom, $F = A_\omega$. ☐

We call $A_\omega = \bigcap_{B \in \mathcal{F}_\omega} B$ *atom of* $\mathcal{F}$ *containing* ω. We then use the next lemma to prove the following theorem.

Lemma 3.2. [20] Let $(\Omega, \mathcal{F})$ be a measurable space. Then $\forall \omega, \omega' \in \Omega$, $A_\omega = A_{\omega'}$ or $A_\omega \cap A_{\omega'} = \emptyset$.

Theorem 3.3. *Let* $(\Omega, \mathcal{F})$ *be a measurable space. Then* A_ω *is an* $\mathcal{F}$*-atom.*

Proof. Note that $\emptyset \neq A_\omega \in \mathcal{F}$. Suppose on the contrary that A_ω is not an $\mathcal{F}$-atom. Then there exists $\emptyset \neq F \in \mathcal{F}$ such that $F \subsetneq A_\omega$. If $\omega \in F$, then $F \in \mathcal{F}_\omega$. So, $A_\omega = \bigcap_{B \in \mathcal{F}_\omega} B \subseteq F$. This implies that $F = A_\omega$, a contradiction. If $\omega \notin F$, then $F \in \mathcal{F} \setminus \mathcal{F}_\omega$. Since $F \neq \emptyset$, there exists $\omega' \in F$ such that $\omega' \neq \omega$. Since $F \subsetneq A_\omega$, $\omega' \in A_\omega$. This implies that $A_{\omega'} \subseteq A_\omega$. Hence, by 3.2, $A_{\omega'} = A_\omega$. Since $\omega' \in F$, by previous argument, $F = A_{\omega'} = A_\omega$, a contradiction. Therefore, A_ω is an $\mathcal{F}$-atom. ☐

Lemma 3.4. *Let* $(\Omega, \mathcal{F})$ *be a measurable space,* A *be an* $\mathcal{F}$*-atom,* $\emptyset \neq B \in \mathcal{F}$ *and* $A \cap B \neq \emptyset$*. Then* $A \subseteq B$.

Proof. Since $A \cap B \neq \emptyset$, there exists $\omega \in \Omega$ such that $\omega \in A$ and $\omega \in B$. By 3.1, $A = A_\omega$. In addition, $B \in \mathcal{F}_\omega$. Thus, $A = A_\omega \subseteq B$. ☐

Theorem 3.5. *Let* $(\Omega, \mathcal{F})$ *be a measurable space.* $\mathcal{F}$ *is atomic if and only if for every element* $\omega \in \Omega$*, there is an* $\mathcal{F}$*-atom containing* ω.

Proof. Let $\mathcal{F}$ be atomic, $\mathcal{S}$ be a collection of atoms of $\mathcal{F}$ and $\omega \in \Omega$. Then $\omega \in \Omega = \bigcup_{A \in \mathcal{S}} A$. This implies that there exist an atom $B \in \mathcal{S}$, of $\mathcal{F}$, such that $\omega \in B$. Now, suppose that for every element $\omega \in \Omega$, there is an $\mathcal{F}$-atom containing ω. Note that $\bigcup_{A \in \mathcal{S}} A \subseteq \Omega$. Let $o \in \Omega$. By assumption, there exists an atom $A(o) \in \mathcal{S}$ containing o. By 3.2, this atom containing o is unique. Thus, $o \in \bigcup_{A \in \mathcal{S}} A$ and $\Omega \subseteq \bigcup_{A \in \mathcal{S}} A$. Hence, $\Omega = \bigcup_{A \in \mathcal{S}} A$ and it follows that $\mathcal{F}$ is atomic. ☐

Theorem 3.6. *Let* $(\Omega, \mathcal{F})$ *be a measurable space. A non-empty set* $A \in \mathcal{F}$ *is an atom if and only if it cannot be written as the union of two non-empty, distinct, elements of* $\mathcal{F}$.

Proof. Let A be an $\mathcal{F}$-atom and suppose on the contrary that there exist distinct, non empty measurable sets, $B, C \in \mathcal{F}$ such that $A = B \cup C$. Then $B \subseteq A$ and $C \subseteq A$. Since A is an atom, it follows that $B = A$ or $B = \emptyset$ and $C = A$ or $C = \emptyset$. By assumption, B, C are non-empty. Hence, $A = B = C$, a contradiction.

Now, suppose that a non-empty set $A \in \mathcal{F}$ cannot be written as the union of two non-empty, distinct, elements of $\mathcal{F}$. Let $D \subseteq A$. Then $A = A \cup D$. By assumption, it follows that $D = \emptyset$ or $D = A$. Thus, A is an $\mathcal{F}$-atom. ☐

Lemma 3.7. *Let* μ *be a nonnegative,* σ*-additive,* σ*-finite measure on* $(\Omega, \mathcal{F})$*. If* A *is an* $\mathcal{F}$*-atom with* $\mu(A) > 0$*, then* A *is also a* μ*-atom.*

Proof. Let A be an $\mathcal{F}$-atom with $\mu(A) > 0$. Then for all $B \in \mathcal{F}$ such that $B \subseteq A$, $B = A$ or $B = \emptyset$. Thus $\mu(B) = \mu(A)$ or $\mu(B) = \mu(\emptyset) = 0$. Hence, A is a μ-atom. $\square$

Remark 3.8. The converse of Lemma 3.7 is not necessarily true.

Example 3.9. Let $\Omega = \{1, 2, 3\}$ and $2^\Omega = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \Omega\}$, the power set of Ω. Define a set function $\mu : \mathcal{F} \to [0, \infty)$ as follows:

$$\mu(A) = \begin{cases} 0, & \text{if } A = \emptyset \text{ or } A = \{3\}, \\ 1, & \text{if } A = \{1\}, A = \{2\}, A = \{1,3\}, A = \{2,3\}, \\ 2, & \text{if } A = \{1,2\}, A = \{1,2,3\}. \end{cases}$$

Notice that $\mu(\emptyset) = 0$. In addition, consider pairwise disjoint subsets $A_1 = \{1\}, A_2 = \{2\}$, $A_3 = \{3\}$.

$$\mu\left(\bigcup_{i=1}^{3} A_i\right) = \mu(\{1, 2, 3\}) = 2,$$

and

$$\sum_{i=1}^{3} \mu(A_i) = \mu(\{1\}) + \mu(\{2\}) + \mu(\{3\}) = 1 + 1 + 0 = 2.$$

Thus, σ-additivity holds. To show σ-finiteness, note that $\Omega = \{1\} \cup \{2\} \cup \{3\}$, where $\{1\}, \{2\}, \{3\} \in \mathcal{F}$ and $\mu(\{1\}), \mu(\{2\}), \mu(\{3\})$ are all finite. Thus, μ is a non negative, σ-finite, measure on $(\Omega, \mathcal{F})$. Notice that $\{3\}$ is an $\mathcal{F}$-atom but not a μ-atom and $\{1, 3\}$ is a μ-atom but not an $\mathcal{F}$-atom.

Proposition 3.10. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $\mathcal{A}$ be the collection of distinct $\mathcal{F}$-atoms and $\mathcal{M}$ be the collection of distinct μ-atoms. $\mathcal{A} = \mathcal{M}$ if and only if all of the following conditions are satisfied:*

(i) *For all $A \in \mathcal{A}$, $\mu(A) > 0$.*

(ii) *$\mathcal{M}$ is pairwise disjoint.*

(iii) *If $B \in \mathcal{F}$ such that $B \subsetneq M$, $M \in \mathcal{M}$, with $\mu(B) = 0$, then $B = \emptyset$.*

Proof. Let $\mathcal{A} = \mathcal{M}$. By 3.7, for all $A \in \mathcal{A}$, $\mu(A) > 0$ and by 3.2, $\mathcal{M}$ is pairwise disjoint. Let $B \in \mathcal{F}$ such that $B \subsetneq M$, where $M \in \mathcal{M}$, with $\mu(B) = 0$ and suppose on the contrary that $B \neq \emptyset$. Then $B \neq \emptyset$, $B \subsetneq M \in \mathcal{A}$, a contradiction. Hence, $B = \emptyset$.

Now, let $\mu(A) > 0$ for all $A \in \mathcal{A}$, $\mathcal{M}$ be pairwise disjoint, and if $B \in \mathcal{F}$ such that $B \subsetneq M$, $M \in \mathcal{M}$, with $\mu(B) = 0$, then $B = \emptyset$. By Result 2.3, it follows that $\mathcal{A} \subseteq \mathcal{M}$. Let $M \in \mathcal{M}$ and $B \in \mathcal{F}$ such that $B \subseteq M$. Then $\mu(B) = 0$ or $\mu(B) = \mu(M)$. If $\mu(B) = 0$, then $B \subsetneq M$, since $\mu(M) > 0$. Thus by assumption, $B = \emptyset$. On the other hand, if $\mu(B) = \mu(M)$, then $B \neq \emptyset$. Suppose that $B \subsetneq M$. This implies that $B \in \mathcal{M}$ since, if $C \in \mathcal{F}$ such that $C \subseteq B \subsetneq M$, $\mu(C) = 0$ or $\mu(C) = \mu(B) = \mu(M)$. Thus, $B, M \in \mathcal{M}$ such that $B \subsetneq M$. A contradiction to the assumption that $\mathcal{M}$ is pairwise disjoint. Hence, $B = M$. Thus, $M \in \mathcal{A}$ and $\mathcal{M} \subseteq \mathcal{A}$. Therefore, it follows that $\mathcal{M} = \mathcal{A}$. $\square$

Theorem 3.11. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\mathcal{F}$ be atomic. Then there exist a μ-atom in $\mathcal{F}$ if and only if there exists an $\mathcal{F}$-atom $A \in \mathcal{F}$ such that $\mu(A) > 0$.*

Proof. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\mathcal{F}$ be atomic. Then $\Omega = \bigcup_{i\in I} A_i$ such that A_i is an $\mathcal{F}$-atom for all $i \in I$. If $i = 1$, then $\Omega = A_1$, $\mathcal{F} = \{\emptyset, \Omega\}$, and we are done. Suppose that $i \geq 2$. Then by 3.2, $(A_i)_{i\in I}$ is pairwise disjoint. Let $M \in \mathcal{F}$ be a μ-atom and suppose on the contrary that for all $\mathcal{F}$-atom $A \in \mathcal{F}$, $\mu(A) = 0$. Then $M \neq \emptyset$ and $\mu(M) > 0$. Since $M \in \mathcal{F}$, $M \subseteq \Omega = \bigcup_{i\in I} A_i$. Moreover, since $(A_i)_{i\in I}$ is pairwise disjoint, there exists $j \in I$ such that A_j is an $\mathcal{F}$-atom and $M \subset A_j$. This implies that $\mu(M) \leq \mu(A_j) = 0$, a contradiction. Thus, there exists an $\mathcal{F}$-atom $A \in \mathcal{F}$ such that $\mu(A) > 0$.

Now, let B be an $\mathcal{F}$-atom such that $\mu(B) > 0$. Then by 3.7, B is a μ-atom in $\mathcal{F}$. □

Theorem 3.12. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\mathcal{F}$ be atomic. If M is a μ-atom, then M is an $\mathcal{F}$-atom.*

Proof. Let $\mathcal{F}$ be atomic and M a μ-atom. Then $M \subseteq \Omega = \bigcup_{i\in I} A_i$ such that A_i is an $\mathcal{F}$-atom for all $i \in I$. By 3.2, $(A_i)_{i\in I}$ is pairwise disjoint. Thus, there exists a unique $k \in I$ such that $M \subseteq A_k$. Since $M \neq \emptyset$ and A_k is an $\mathcal{F}$-atom it follows that $M = A_k$ and M is an $\mathcal{F}$-atom. □

Theorem 3.13. *Let $(\Omega, \mathcal{F})$ be a measurable space, and $\{C_i \mid i \in \mathbb{N}\} \subseteq \mathcal{F}$.*

(i) *If $\bigcup\limits_{i=1}^{+\infty} C_i \neq \emptyset$, then $\bigcup\limits_{i=1}^{+\infty} C_i$ is an $\mathcal{F}$-atom if and only if there exist $i \in \mathbb{N}$ such that C_i is an $\mathcal{F}$-atom and $C_i = C_j$ for all $j \in \mathbb{N}$ such that $C_j \neq \emptyset$ and $j \neq i$.*

(ii) *If $\bigcap\limits_{i=1}^{+\infty} C_i \neq \emptyset$, then $\bigcap\limits_{i=1}^{+\infty} C_i$ is an $\mathcal{F}$-atom if and only if there exists a unique $\emptyset \neq A \in \mathcal{F}$ such that A is an $\mathcal{F}$-atom and $A \subseteq C_i$ for all $i \in \mathbb{N}$.*

Proof. Let $F_\sigma := \bigcup\limits_{i=1}^{+\infty} C_i$ be non-empty and an $\mathcal{F}$-atom. Proceeding by contradiction, suppose that for all $i \in \mathbb{N}$, C_i is not an $\mathcal{F}$-atom. Since $F_\sigma \neq \emptyset$, there exists $k \in \mathbb{N}$ such that $\emptyset \neq C_k \in \mathcal{F}$ and C_k is not an $\mathcal{F}$-atom. Hence, there exists $\emptyset \neq A \in \mathcal{F}$ such that $A \subsetneq C_k \subseteq \bigcup\limits_{i=1}^{+\infty} C_i = F_\sigma$, a contradiction. On the other hand, suppose there exists $j \in \mathbb{N}$ such that for $j \neq i$, $C_j \neq \emptyset$, and $C_j \neq C_i$. Then $C_j \in \mathcal{F}$ and $\emptyset \neq C_j \subsetneq \bigcup\limits_{i=1}^{+\infty} C_i = F_\sigma$, a contradiction. Conversely, suppose that there exist $i \in \mathbb{N}$ such that C_i is an $\mathcal{F}$-atom and $C_i = C_j$ for all $j \in \mathbb{N}$ such that $C_j \neq \emptyset$ and $j \neq i$. Then $F_\sigma = \bigcup\limits_{i=1}^{+\infty} C_i$ is non-empty and $F_\sigma = C_i$. It follows that $F_\sigma = \bigcup\limits_{i=1}^{+\infty} C_i$ is an $\mathcal{F}$-atom, and hence, condition (i) holds.

Now, let $G_\delta := \bigcap\limits_{i=1}^{+\infty} C_i$ be an $\mathcal{F}$-atom. Then $G_\delta \subseteq C_i$ for all $i \in \mathbb{N}$. Suppose there exists $\emptyset \neq C \in \mathcal{F}$ such that $C \subseteq C_i$ for all $i \in \mathbb{N}$. Then $C \subseteq \bigcap\limits_{i=1}^{+\infty} C_i = G_\delta$. Since G_δ is an $\mathcal{F}$-atom, it follows that $C = G_\delta$. Conversely, let $G_\delta = \bigcap\limits_{i=1}^{+\infty} C_i$ be non-empty and suppose that there exists a unique $\emptyset \neq A \in \mathcal{F}$ such that A is an $\mathcal{F}$-atom and $A \subseteq C_i$ for all $i \in \mathbb{N}$. Note that $G_\delta \in \mathcal{F}$ and $G_\delta = \bigcap\limits_{i=1}^{+\infty} C_i \subseteq C_i$ for all $i \in \mathbb{N}$. By assumption, $A = G_\delta$ and it follows that $G_\delta = \bigcap\limits_{i=1}^{+\infty} C_i$ is an $\mathcal{F}$-atom. Hence, (ii) holds. □

4 Atoms of Trace σ-Algebra and Pre-image σ-Algebra

This section examines the concepts of atoms in trace σ-algebras and pre-image σ-algebras. The definition of trace σ-algebra is defined in [18] as follows; Let $E \subset \Omega$ be any set and let $\mathcal{F}$ be σ-algebra on Ω. Then $\mathcal{F}_E := \{E \cap F \mid F \in \mathcal{F}\}$ is a σ-algebra on E, called the *trace σ-algebra* on E.

Theorem 4.1. *Suppose $Y \subseteq \Omega$, $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{B} = \{Y \cap A : A \in \mathcal{F}\}$ be the Trace σ-algebra on Y. If A is an $\mathcal{F}$-atom then $Y \cap A$ is a $\mathcal{B}$-atom.*

Proof. Let A be an $\mathcal{F}$-atom and $\emptyset \neq A' = Y \cap A \in \mathcal{B}$. Let $B' \in \mathcal{B}$ and $B' \subsetneq A'$. If $B' = \emptyset$ then we are done. Suppose $B' \neq \emptyset$. Then there exists $\emptyset \neq B \in \mathcal{F}$ such that $B' = Y \cap B \neq \emptyset$. In addition, $B' = Y \cap B \subsetneq A' = Y \cap A$. This implies that $B' \subseteq A$. Suppose $B' \in \mathcal{F}$. Since A is an $\mathcal{F}$-atom, $B' = A$. Hence, $A' = Y \cap A = Y \cap B' = Y \cap (Y \cap B) = Y \cap B = B'$.

Now, Let $B' \notin \mathcal{F}$. Note that $B' = Y \cap B \subseteq A$. This means that $A \cap B \neq \emptyset$. By 3.4, $A \subseteq B$. Thus, $A' = Y \cap A \subseteq Y \cap B = B'$ and it follows that $A' = B'$. Therefore, A' is an $\mathcal{F}$-atom. ☐

The definition of Pre-image σ-Algebra is defined in [18] as follows; Let Ω and Y be nonempty sets, $f : \Omega \to Y$ be a map and let $\mathcal{F}$ be a σ-algebra on Y. Then $f^{-1}(\mathcal{F}) := \{f^{-1}(F) \mid F \in \mathcal{F}\}$ is a σ-algebra on Ω, called the *pre-image σ-algebra*.

Theorem 4.2. *Let f be a function from a set Ω to a set Y, $\mathcal{F}$ be a σ-algebra on Y, and $\sigma(f) = \{f^{-1}(F) \mid F \in \mathcal{F}\}$ be the σ-algebra on Ω. A nonempty $A' \in \sigma(f)$ is a $\sigma(f)$-atom if there exists an $\mathcal{F}$-atom A such that $f^{-1}(A) = A'$.*

Proof. Let f be a function from a set Ω to a set Y, $\mathcal{F}$ be a σ-algebra on Y, $\sigma(f) = \{f^{-1}(F) \mid F \in \mathcal{F}\}$ be the σ-algebra on Ω and let C be an $\mathcal{F}$-atom and $\emptyset \neq f^{-1}(C) = C' \in \sigma(f)$. Let $D' \in \sigma(f)$ such that $D' \subseteq C'$. Then there exists $D \in \mathcal{F}$ such that $f^{-1}(D) = D' \subseteq C' = f^{-1}(C)$. If $D = \emptyset$, then $D' = f^{-1}(D) = \emptyset$ and we are done. Suppose $D \neq \emptyset$. If $f^{-1}(D) = \emptyset$, then we are also done. Suppose $D \neq \emptyset$ and $f^{-1}(D) = D' \neq \emptyset$. Since $\emptyset \neq f^{-1}(C)$, $C \cap D \neq \emptyset$. By Lemma 3.4, $C \subseteq D$. Thus, $f^{-1}(C) \subseteq f^{-1}(D)$ and it follows that $C' \subseteq D'$. Hence, $D' = C'$ which implies that C' is an $\sigma(f)$-atom. ☐

Theorem 4.3. *The converse of Theorem 4.2 holds if $A \subseteq f(\Omega)$.*

Proof. Let F' be a $\sigma(f)$-atom. Then $\emptyset \neq F' \in \sigma(f)$ and there exists $\emptyset \neq F \in \mathcal{F}$ such that $F' = f^{-1}(F)$. Suppose $F \subseteq f(\Omega)$ and let $G \in \mathcal{F}$ such that $G \subseteq F$. Then $f^{-1}(G) \subseteq f^{-1}(F) = F'$ and $f^{-1}(G) \in \sigma(f)$. This implies that $f^{-1}(G) = \emptyset$ or $f^{-1}(G) = f^{-1}(F)$. If $f^{-1}(G) = \emptyset$. Then $G \cap f(\Omega) = \emptyset$. Hence, $G \subseteq (Y \setminus f(\Omega))$. Note that $F \subseteq f(\Omega)$. Thus, $G \subseteq F \cap (Y \setminus f(\Omega)) = \emptyset$ and it follows that $G = \emptyset$. Suppose $f^{-1}(G) = f^{-1}(F)$. Then $f^{-1}(F \setminus G) = f^{-1}(F) \setminus f^{-1}(G) = \emptyset$. This implies that $F \setminus G \subseteq Y \setminus f(\Omega)$. Notice that $F \setminus G \subseteq F \subseteq f(\Omega)$. Hence, $F \setminus G \subseteq f(\Omega) \cap (Y \setminus f(\Omega)) = \emptyset$. Thus, $F = G$. Therefore, F is an $\mathcal{F}$-atom. ☐

5 Atoms of Product σ-Algebra

This section examines the concepts of atoms in product σ-algebras. But before that we introduce the notion of quarks in set systems, examine their fundamental properties and their relationship to σ-algebras. Establishing these results are essential, as quarks will play a key role in characterizing atoms within product σ-algebras.

Definition 5.1. Let $\mathcal{S}$ be a set system of a set Ω. A non-empty set $Q \in \mathcal{S}$ is a *quark* if for all $S \in \mathcal{S}$, $S \subsetneq Q$, implies $S = \emptyset$.

Example 5.2. The following are examples of set systems with corresponding quarks if any.

(i) [19] Let $\Omega = \{1, 2, \ldots, 2k\}$ for some fixed $k \in \mathbb{N}$, and consider the Dynkin system $\mathcal{D} = \{A \subseteq X \mid \#A \text{ is even}\}$. Then $\{1, 2\}$ is a quark of $\mathcal{D}$.

(ii) Let Ω be a nonempty set. Consider $\mathcal{S} = \{\emptyset\} \subseteq 2^{\Omega}$. Then $\mathcal{S}$ is a set system but in this case, $\mathcal{S}$ doesn't have a quark.

(iii) [19] Let $\Omega = [0, 1]$, and let $\mathcal{F}$ be the σ-algebra generated by the intervals $(0, \frac{1}{4})$ and $(\frac{3}{4}, 1]$. Then,

$$\mathcal{F} = \left\{\emptyset, [0, \tfrac{1}{4}), [\tfrac{1}{4}, \tfrac{3}{4}], (\tfrac{3}{4}, 1], [0, \tfrac{3}{4}], [\tfrac{1}{4}, 1], [0, \tfrac{1}{4}) \cup (\tfrac{3}{4}, 1], [0, 1]\right\}.$$

Note that $\mathcal{F}$ is also a set system. Thus, the quarks of $\mathcal{F}$ are $[0, \frac{1}{4})$, $[\frac{1}{4}, \frac{3}{4}]$, $(\frac{3}{4}, 1]$. In this case, these quarks are precisely the atoms of $\mathcal{F}$.

Theorem 5.3. *Let $(\Omega, \mathcal{F})$ be a measurable space. Q is a quark of $\mathcal{F}$ if and only if Q is an $\mathcal{F}$-atom.*

Proof. Let $\mathcal{F}$ be a σ-algerba on Ω and Q be a quark of $\mathcal{F}$. Then $\emptyset \neq Q \in \mathcal{F}$ such that for all $F \in \mathcal{F}$, $F \subsetneq Q$, $F = \emptyset$. Thus Q is an $\mathcal{F}$-atom. Now, since $\mathcal{F}$ is a σ-algerba on Ω, $\emptyset \in \mathcal{F}$ and it follows that $\mathcal{F}$ is a set system on Ω. Let Q be an $\mathcal{F}$-atom. then $\emptyset \neq Q \in \mathcal{F}$ such that for all $F \in \mathcal{F}$, $F \subsetneq Q$, $F = \emptyset$. Thus Q is a quark of $\mathcal{F}$. ☐

Theorem 5.4. *Let $\mathcal{S}$ be a set system of a set Ω that is closed under intersection. If Q and R are quarks of $\mathcal{S}$, then either $Q = R$ or $Q \cap R = \emptyset$.*

Proof. Since $\mathcal{S}$ is closed under intersections and $Q, R \in \mathcal{S}$, it follows that $Q \cap R \in \mathcal{S}$. Suppose $Q \cap R \neq \emptyset$. Then $Q \cap R \subseteq Q$, and since Q is a quark, we must have $Q \cap R = Q$. Similarly, $Q \cap R \subseteq R$, and since R is a quark, it follows that $Q \cap R = R$. Therefore, $Q = R$. Hence, either $Q = R$ or $Q \cap R = \emptyset$. ☐

We introduce the following notation to be used in subsequent proofs: Let $\omega \in \Omega$. Define $\mathcal{S}_\omega = \{B \in \mathcal{S} \mid \omega \in B\}$ and $Q_{\mathcal{S}}(\omega) = \bigcap_{B \in \mathcal{S}_\omega} B$. Throughout the remainder of the paper, we assume that $\mathcal{S}_\omega \neq \emptyset$ for all $\omega \in \Omega$.

Theorem 5.5. *Let $\mathcal{S}$ be a set system of a set Ω that is closed under intersection. Then $Q_{\mathcal{S}}(\omega)$ is a quark of $\mathcal{S}$.*

Proof. Since $\mathcal{S}$ is closed under intersection, $Q_{\mathcal{S}}(\omega) \in \mathcal{S}$. Let $S \in \mathcal{S}$ such that $S \subsetneq Q_{\mathcal{S}}(\omega)$. If $\omega \in S$, then $S \in \mathcal{S}_\omega$. So, $Q_{\mathcal{S}}(\omega) = \bigcap_{B \in \mathcal{S}_\omega} B \subseteq S$. This implies that $S = Q_{\mathcal{S}}(\omega)$, a contradiction. If $\omega \notin S$, then $S \in \mathcal{S} \setminus \mathcal{S}_\omega$. Suppose $S \neq \emptyset$. Then there exists $\omega' \in S$ such that $\omega' \neq \omega$. Since $S \subsetneq Q_{\mathcal{S}}(\omega)$, $\omega' \in Q_{\mathcal{S}}(\omega)$. This implies that $Q_{\mathcal{S}}(\omega') \subseteq Q_{\mathcal{S}}(\omega)$. Hence, $Q_{\mathcal{S}}(\omega') = Q_{\mathcal{S}}(\omega)$. Since $\omega' \in S$, by previous argument, $S = Q_{\mathcal{S}}(\omega') = Q_{\mathcal{S}}(\omega)$, a contradiction. Therefore, $S = \emptyset$. ☐

Theorem 5.6. *Let $\mathcal{S}$ be a set system of a set Ω that is closed under intersection. If Q is a quark containing ω, then $Q = Q_{\mathcal{S}}(\omega)$.*

Proof. Since $Q \in \mathcal{S}$ such that $\omega \in Q$, $Q_{\mathcal{S}}(\omega) = \bigcap_{B \in \mathcal{S}_\omega} B \subseteq Q$. Note that $\emptyset \neq Q_{\mathcal{S}}(\omega) \in \mathcal{S}$, $\mathcal{S}$ being closed under intersection, and Q is a quark. Hence, $Q = Q_{\mathcal{S}}(\omega)$. ☐

We now state a lemma that will also be used in the theorem that follows.

Lemma 5.7. [6] ($\pi - \delta$ Theorem) If $\mathcal{P}$ is a π-system and $\mathcal{D}$ is a Dynkin-system that contains $\mathcal{P}$, then $\sigma(\mathcal{P}) \subset \mathcal{D}$.

Theorem 5.8. *Let $\mathcal{P}$ be a π-system and $\mathcal{D}$ be a Dynkin system where $\mathcal{P} \subseteq \mathcal{D}$. If Q is a quark of $\mathcal{D}$ such that $Q \in \mathcal{P}$, then there exist a σ-algebra $\mathcal{F}$ such that Q is an $\mathcal{F}$-atom.*

Proof. Let $\mathcal{P}$ be a π-system, $\mathcal{D}$ be a Dynkin system where $\mathcal{P} \subseteq \mathcal{D}$, and Q be a quark of $\mathcal{D}$ such that $Q \in \mathcal{P}$. By 5.7, $\mathcal{P} \subseteq \sigma(\mathcal{P}) \subseteq \mathcal{D}$. This means that $Q \in \sigma(\mathcal{P})$. Suppose on the contrary that Q is not an atom of $\sigma(\mathcal{P})$. Then there exists a non-empty $S \in \sigma(\mathcal{P})$ such that $S \subsetneq Q$. Since $\sigma(\mathcal{P}) \subseteq \mathcal{D}$, $S \in \mathcal{D}$, which implies that Q is not a quark of $\mathcal{D}$, a contradiction. Thus, Q is a $\sigma(\mathcal{P})$-atom. Therefore, the conclusion holds. ☐

Corollary 5.9. *If a quark $Q \in \mathcal{D}$ satisfies the properties of 5.8, then Q is also a quark of $\mathcal{P}$ if and only if $\emptyset \in \mathcal{P}$.*

Proof. Let $Q \in \mathcal{D}$ be a quark that satisfies the properties of 5.8 and suppose $\emptyset \in \mathcal{P}$. Then $\mathcal{P}$ is a set system and the argument proceeds in the same manner as the proof of Proposition 5.8, substituting $\sigma(\mathcal{P})$ and $\mathcal{P}$ with the appropriate counterparts. Now, let Q be also a quark of $\mathcal{P}$. Then $\mathcal{P}$ is a set system and $\emptyset \in \mathcal{P}$. ☐

We now define Product σ-Algebra as in [18] as follows; Let $(X, \mathcal{J})$ and $(Y, \mathcal{H})$ be two measurable spaces. The *product σ-algebra* on $X \times Y$, denoted by $\mathcal{J} \otimes \mathcal{H}$, is the σ-algebra generated by $\mathcal{J} \times \mathcal{H} = \{J \times H \mid J \in \mathcal{J},\ H \in \mathcal{H}\}$. That is, $\mathcal{J} \otimes \mathcal{H} := \sigma\left(\{J \times H : J \in \mathcal{J},\ H \in \mathcal{H}\}\right)$. The measurable space $(X \times Y, \mathcal{J} \otimes \mathcal{H})$ is called the *product of the measurable spaces*. The following lemma will be used in the theorem that follows.

Lemma 5.10. [18] Let $(X, \mathcal{J})$,$(Y, \mathcal{H})$ be two measurable spaces, and $\mathcal{V} = \{J \times H : J \in \mathcal{J}, H \in \mathcal{H}\}$. Then $\mathcal{V}$ is closed under intersection and $\emptyset \in \mathcal{V}$.

Theorem 5.11. *Let $(X, \mathcal{J})$,$(Y, \mathcal{H})$ be two measurable spaces, and $\mathcal{V} = \{J \times H : J \in \mathcal{J}, H \in \mathcal{H}\}$. Then $\mathcal{V}$ is a set system on $X \times Y$. Moreover, A non-empty $Q \in \mathcal{V}$ is an quark of $\mathcal{V}$ if and only if there exists an atom $Q_1 \in \mathcal{J}$ and $Q_2 \in \mathcal{H}$ such that $Q = Q_1 \times Q_2$.*

Proof. By 5.10, $\mathcal{V}$ is a set system on $X \times Y$. Let Q_1 be a $\mathcal{J}$-atom and Q_2 be an $\mathcal{H}$-atom such that $Q = Q_1 \times Q_2$. Then $Q_1 \neq \emptyset$, $Q_2 \neq \emptyset$, and $\emptyset \neq Q \in \mathcal{V}$. Suppose on the contrary that Q is not a quark of $\mathcal{V}$. Since $V \neq \emptyset$, there exists non-empty sets $V_1 \in \mathcal{J}$ and $V_2 \in \mathcal{H}$ such that $V = V_1 \times V_2$. Thus, $V_1 \times V_2 \subsetneq Q$. This implies that $V_1 \subsetneq Q_1$ or $V_2 \subsetneq Q_2$, a contradiction. Hence, Q is a quark of $\mathcal{V}$.

Assume that Q is a quark of $\mathcal{V}$. Then $\emptyset \neq Q \in \mathcal{V}$ and there exists nonempty sets $Q_1 \in \mathcal{J}$ and $Q_2 \in \mathcal{H}$ such that $Q = Q_1 \times Q_2$. Suppose on the contrary that either Q_1 is not a $\mathcal{J}$-atom or Q_2 is not an $\mathcal{H}$-atom. If Q_1 is not a $\mathcal{J}$-atom, then there exists $\emptyset \neq R \in \mathcal{J}$ such that $R \subsetneq Q_1$. Consider $R \times Q_2$. Notice that $\emptyset \neq R \times Q_2 \in \mathcal{V}$ and $R \times Q_2 \subsetneq Q$, a contradiction. A similar argument holds if Q_2 is not $\mathcal{H}$-atom. Therefore, the claim holds. ☐

We now state a result which will be used in the following theorem.

Lemma 5.12. [20] Let $\mathcal{C} \subseteq 2^{\Omega}$, $\mathcal{F} = \sigma(\mathcal{C})$ and $A_{\mathcal{F}}(\omega)$ the atom of $\mathcal{F}$ containing ω. $\forall \omega \in \Omega$, define $\mathcal{C}_\omega = \{B \in \mathcal{C} \mid \omega \in B\}$ and $A_{\mathcal{C}}(\omega) = \bigcap_{B \in \mathcal{C}_\omega} B$. If the generator $\mathcal{C}$ satisfies the property that $\forall A \in \mathcal{C}$, $A^c \in \kappa(\mathcal{C})$, then $A_{\mathcal{F}}(\omega) = A_{\mathcal{C}}(\omega)$.

Theorem 5.13. *Let $(X, \mathcal{J})$,$(Y, \mathcal{H})$ be two measurable spaces, $\mathcal{V} = \{J \times H : J \in \mathcal{J}, H \in \mathcal{H}\}$ and $\mathcal{J} \otimes \mathcal{H} = \sigma(\mathcal{V})$ be the product σ-algebra on the product space $X \times Y$. S is a $\mathcal{J} \otimes \mathcal{H}$-atom if and only if S is a quark of $\mathcal{V}$.*

Proof. To prove this theorem we need to show that a $\mathcal{J}\otimes\mathcal{H}$-atom, S, that contains $x\in X\times Y$ is equal to a quark, Q, in $\mathcal{V}$ that also contains $x\in X\times Y$. Notice that by 5.10, $\mathcal{V}$ is closed under intersection and is a set system on $X\times Y$. Hence, 5.5 and 5.6 holds. Let $V\in\mathcal{V}$. Then $V=J\times H$ such that $J\in\mathcal{J}$, $H\in\mathcal{H}$. Since $\mathcal{J}$ and $\mathcal{H}$ are σ-algebra on X and Y respectively, it follows that $J^c\in\mathcal{J}, H^c\in\mathcal{H}$. Hence, $(J^c\times H^c),(J^c\times H),(J\times H^c)\in\mathcal{V}\subseteq\kappa(\mathcal{V})$. Now, since $\kappa(\mathcal{V})$ is closed under countable union, it follows that $V^c=(J\times H)^c=(J^c\times H^c)\cup(J^c\times H)\cup(J\times H^c)\in\kappa(\mathcal{V})$. Thus, by 5.12, our claim holds. $\square$

Summary and Recommendations

In this study, we discussed the concepts of μ-atoms and $\mathcal{F}$-atoms and examined their relationship within a measure space $(\Omega,\mathcal{F},\mu)$. We identified conditions under which the $\mathcal{F}$-atoms coincide with μ-atoms. We also characterized $\mathcal{F}$-atoms in trace, pre-image, and product σ-algebras. Moreover, we introduced the notion of a quark and explored its properties. These results contribute to a deeper understanding of the structure of set systems and their minimal elements.

For future directions, it is recommended to construct explicit examples of set systems where neither $\mathcal{F}$-atoms nor μ-atoms exist, to better understand when atoms do not exist in more general settings. Additionally, formulating a counterpart to the μ-atom concept within a set system could provide a broader framework for analyzing minimal elements in more general setting.

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