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On the AB-Generalized Lucas Sequence by Hessenberg Permanents and Determinants

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Abstract: In this paper, we give some representations of the terms of the AB-generalized Lucas sequence $\{v_n\}$ using Hessenberg matrices. These formulas generalize some results obtained by Kiliç et al. in [12].

Keywords/Phrases: Fibonacci sequence, Lucas sequence, generalized Fibonacci sequence, generalized Lucas sequence, AB-generalized Fibonacci sequence, AB-generalized Lucas sequence

1 Introduction

For $n > 0$, the Fibonacci sequence $\{F_n\}$ is defined by

$$F_{n+1} = F_n + F_{n-1},$$

where $F_0 = 0$ and $F_1 = 1$. The Lucas sequence $\{L_n\}$ is defined by

$$L_{n+1} = L_n + L_{n-1},$$

where $L_0 = 2$ and $L_1 = A$.

In [5], Kiliç introduced the generalized Fibonacci sequence and gave the explicit formulas for the sums of the terms of these sequences using matrix methods. He constructed essential generating matrices and used matrix properties to obtain these sums. Kiliç's definition provided a motivation to the construction of the AB-generalized Fibonacci sequence and AB-generalized Lucas sequence.

Let $n > 0$ and let A and B be nonzero integers with $A^2 + 4B \neq 0$. The AB-**generalized Fibonacci sequence** $\{u_n\}$ has the recurrence relation

$$u_{n+1} = Au_n + Bu_{n-1},$$

where $u_0 = 0$ and $u_1 = 1$. The AB-**generalized Lucas sequence** $\{v_n\}$ has the recurrence relation

$$v_{n+1} = Av_n + Bv_{n-1},$$

where $v_0 = 2$ and $v_1 = A$.

In [1], the author gave the combinatorial representations of $\{u_n\}$ and $\{v_n\}$ and are respectively given by

$$u_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} A^{n-2k} B^k,$$

$$v_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} A^{n-2k} B^k.$$

Numerous authors have been interested with the second order linear recurrences and show their relationships between the permanents and determinants of tridiagonal matrices.

In [10], the authors gave interesting results involving the permanent of the $(-1, 0, 1)$-matrix and the Fibonacci number F_{n+1}. Consequently, the authors established some results involving the positively and negatively subscripted terms of the Fibonacci and Lucas numbers.

In [8], the authors discovered the families of $(0, 1)$-matrices and then gave the relationships between the permanents of these matrices and the sums of the Fibonacci and Lucas numbers.

In [4], the author introduced two tridiagonal matrices and then gave the relationships between the permanents and determinants of these matrices and the second order linear recurrences.

In [11], the authors introduced the two generalized doubly stochastic matrices and then show the relationships between the doubly stochastic permanents and the second order linear recurrences.

In [9] and [12], the authors define lower Hessenberg matrices and gave the relationships between the permanents and determinants of these matrices and the generalized Fibonacci, generalized Lucas and Pell sequences.

Analogous to the methods done in [9], the authors in [14] also define lower Hessenberg matrices and gave the relationships between the permanents and determinants of these matrices and the AB-generalized Fibonacci sequence.

A lower Hessenberg matrix $M_n = (a_{ij})$ is an $n \times n$ matrix where $a_{jk} = 0$ whenever $k > j+1$ and $a_{j(j+1)} \neq 0$ for some j. Clearly,

$$M_n = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & \vdots & 0 \\ a_{31} & a_{32} & a_{33} & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{(n-2)(n-1)} & 0 \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(n-2)} & a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n(n-1)} & a_{nn} \end{bmatrix}.$$

In [2], the authors consider the above general lower Hessenberg matrix and then give the following determinant formula. For $n \geq 2$,

$$\det M_n = a_{nn} \cdot \det M_{n-1} + \sum_{r=1}^{n-1} \left((-1)^{n-r} a_{mr} \prod_{j=r}^{n-1} a_{j(j+1)} \det M_{r-1} \right).$$

In this paper, we consider the AB-generalized Lucas sequence $\{v_n\}$ and then we show the relationships between the AB-generalized Lucas sequence and the Hessenberg determinants and permanents. Also, we give the representations of v_{2n} and v_{2n+1}. The authors in [12] proved the results for special case $B = 1$.

2 On the AB-Generalized Lucas Sequence by Hessenberg Matrices

Consider first the following Hessenberg matrices. Let the $n \times n$ lower Hessenberg matrix G_n defined by

$$G_n = \begin{bmatrix} A^2+3B & B & 0 & \cdots & 0 & 0 \\ B & A^2+B & B & \ddots & \vdots & 0 \\ B & B & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & B & 0 \\ B & B & \cdots & B & A^2+B & B \\ B & B & B & \cdots & B & A^2+B \end{bmatrix}.$$

We shall use the following two results later.

Theorem 2.1 [14] *For every $n > 0$,*

$$u_{n+2} = \frac{\det H_n}{A^{n-1}}, \quad \textit{where}$$

$$H_n = \begin{bmatrix} A^2+B & B & 0 & \cdots & 0 & 0 \\ B & A^2+B & B & \ddots & \vdots & 0 \\ B & B & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & B & 0 \\ B & B & \cdots & B & A^2+B & B \\ B & B & B & \cdots & B & A^2+B \end{bmatrix}.$$

By induction, the following result holds.

Lemma 2.2 *For every* $n \geq 0$,

$$v_{n+1} = u_{n+2} + Bu_n.$$

Theorem 2.3 *For every* $n > 0$,

$$v_{n+2} = \frac{\det G_n}{A^{n-2}} \text{ or}$$

$$\det G_n = \sum_{k=0}^{\lfloor \frac{n+2}{2} \rfloor} \frac{n+2}{n+2-k} \binom{n+2-k}{k} A^{2n-2k} B^k.$$

Proof: If $n = 1$, then $\det G_1 = \det\,[A^2 + 3B] = A^{1-2} v_{1+2}$. Now, let $n \geq 2$. By Lemma 2.2, Theorem 2.1 and using the fact that $\det H_n = (A^2 + B)\det H_{n-1} - B^2 \det H_{n-2}$, we have

$$\det G_n = (A^3 + 3B)\det H_{n-1} - B^2 \det H_{n-2} = A^{n-2} v_{n+2},$$

and the proof is complete. □

It is worth noting that if $A = B = 1$, then by Theorem 2.3, we have

$$\begin{vmatrix} 4 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \ddots & \vdots & 0 \\ 1 & 1 & 2 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{vmatrix}_{n \times n} = L_{n+2},$$

where L_n is the n^{th} Lucas number.

As done by the authors in [12], we shall now consider the permanent of a Hessenberg matrix. We define first the following concepts.

A matrix M is said to be *convertible* if there is an $n \times n$ $(1, -1)$-matrix H such that per $M = \det(M \circ H)$, where $M \circ H$ denotes the Hadamard product of M and H. The matrix H is called the *converter* of M.

Now, let S be an $n \times n$ $(1, -1)$-matrix defined by

$$S = \begin{bmatrix} 1 & -1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & -1 & \ddots & \vdots & 1 \\ 1 & 1 & 1 & \ddots & 1 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

We denote the Hadamard product $G_n \circ S$ by D_n. Then

$$D_n = \begin{bmatrix} A^2+3B & -B & 0 & \cdots & 0 & 0 \\ B & A^2+B & -B & \ddots & \vdots & 0 \\ B & B & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -B & 0 \\ B & B & \cdots & B & A^2+B & -B \\ B & B & B & \cdots & B & A^2+B \end{bmatrix}.$$

The following result is a consequence of Theorem 2.3.

Corollary 2.4 *For every $n > 0$,*

$$v_{n+2} = \frac{\text{per } D_n}{A^{n-2}} \text{ or}$$

$$\text{per } D_n = \sum_{k=0}^{\lfloor \frac{n+2}{2} \rfloor} \frac{n+2}{n+2-k} \binom{n+2-k}{k} A^{2n-2k} B^k.$$

3 Representations of v_{2n} and v_{2n+1}

In this section, we give the representations of v_{2n} and v_{2n+1} using permanents and determinants of some Hessenberg matrices.

Firstly, let the $n \times n$ lower Hessenberg matrix E_n defined by

$$E_n = \begin{bmatrix} A^2+3B & -B & 0 & \cdots & 0 & 0 \\ A^2+2B & A^2+B & -B & \ddots & \vdots & 0 \\ A^2+2B & A^2 & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -B & 0 \\ A^2+2B & A^2 & \cdots & A^2 & A^2+B & -B \\ A^2+2B & A^2 & A^2 & \cdots & A^2 & A^2+B \end{bmatrix}.$$

The following two results are needed in obtaining the representations of v_{2n} and v_{2n+1}.

Theorem 3.1 [14] *For every $n > 0$,*

$$u_{2n} = \frac{\det V_n}{A}, \quad \textit{where}$$

$$V_n = \begin{bmatrix} A^2 & -B & 0 & \cdots & 0 & 0 \\ A^2 & A^2+B & -B & \ddots & \vdots & 0 \\ A^2 & A^2 & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -B & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+B & -B \\ A^2 & A^2 & A^2 & \cdots & A^2 & A^2+B \end{bmatrix}.$$

Theorem 3.2 [14] *For every $n > 0$,*

$$u_{2n+1} = \det W_n, \quad \textit{where}$$

$$W_n = \begin{bmatrix} A^2+B & -B & 0 & \cdots & 0 & 0 \\ A^2 & A^2+B & -B & \ddots & \vdots & 0 \\ A^2 & A^2 & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -B & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+B & -B \\ A^2 & A^2 & A^2 & \cdots & A^2 & A^2+B \end{bmatrix}.$$

Theorem 3.3 *For every $n > 0$,*

$$v_{2n+1} = A \det E_n.$$

Proof: The case where $n = 1$ is trivial. Suppose that $v_{2n+1} = A \det E_n$. We now show that the equation also holds for $n + 1$. Now, subtracting the n^{th} row from the $(n+1)^{th}$ row and expanding along the last column, we have $\det E_{n+1} = (A^2 + 2B) \det E_n - B^2 \det E_{n-1}$. By the assumption and the recurrence relation of the sequence $\{v_n\}$, we have $\det E_{n+1} = A^{-1} v_{2n+3}$. Thus, by induction, the assertion must be true. □

Secondly, let the $n \times n$ lower Hessenberg matrix Z_n defined by

$$Z_n = \begin{bmatrix} A^2+2B & -B & 0 & \cdots & 0 & 0 \\ A^2 & A^2+B & -B & \ddots & \vdots & 0 \\ A^2 & A^2 & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -B & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+B & -B \\ A^2 & A^2 & A^2 & \cdots & A^2 & A^2+B \end{bmatrix}.$$

Then we have the following result.

Theorem 3.4 *For every $n > 0$,*

$$v_{2n} = \det Z_n.$$

Proof: The equation holds for $n = 1$. Now assume that $v_{2n} = \det Z_n$. We now show that the equation also holds for $n + 1$. Then expanding along the first row, we have $\det Z_{n+1} = (A^2 + 2B) \det W_n + B \det V_n$. By assumption, recurrence relation of the sequence $\{v_n\}$, Lemma 2.2 and Theorem 3.3, we have $\det Z_{n+1} = v_{2n+2}$. Hence, by induction, the conclusion follows. □

Consider again the $n \times n$ $(1, -1)$ matrix S defined previously. We denote the Hadamard products $E_n \circ S$ and $Z_n \circ S$ by M_n and Y_n, respectively. Then

$$M_n = \begin{bmatrix} A^2+3B & B & 0 & \cdots & 0 & 0 \\ A^2+2B & A^2+B & B & \ddots & \vdots & 0 \\ A^2+2B & A^2 & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & B & 0 \\ A^2+2B & A^2 & \cdots & A^2 & A^2+B & B \\ A^2+2B & A^2 & A^2 & \cdots & A^2 & A^2+B \end{bmatrix}$$

and

$$Y_n = \begin{bmatrix} A^2+2B & B & 0 & \cdots & 0 & 0 \\ A^2 & A^2+B & B & \ddots & \vdots & 0 \\ A^2 & A^2 & A^2+B & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & B & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+B & B \\ A^2 & A^2 & A^2 & \cdots & A^2 & A^2+B \end{bmatrix}.$$

The following results are consequences of Theorems 3.3 and 3.4.

Corollary 3.5 *For every* $n > 0$, $v_{2n+1} = A \operatorname{per} M_n$.

Corollary 3.6 *For every* $n > 0$, $v_{2n} = \operatorname{per} Y_n$.

Using the above results and identity in [1], we have the following representations:

$$\det E_n = \text{per } M_n = \sum_{k=0}^{\lfloor \frac{2n+1}{2} \rfloor} \frac{2n+1}{2n+1-k} \binom{2n+1-k}{k} A^{2n-2k} B^k,$$

and

$$\det Z_n = \text{per } Y_n = \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} A^{2n-2k} B^k.$$

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