



SEMI-DELTA-CONTINUOUS FUNCTIONS IN THE BOX TOPOLOGY AND SOME VERSIONS OF SEPARATION AXIOMS

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Abstract

In this paper, we characterize the concept of δ_s -continuous functions from an arbitrary topological space into the topological space with the box topology. Moreover, we introduce and characterize some versions of separation axioms related to δ_s -open sets.

1 Introduction

In 1963, Norman Levine [18] made the first approach to the study of open sets in topological spaces by introducing the concepts of semi-open sets, semi-closed sets, and semi-continuity. His groundbreaking work provided a foundation that inspired subsequent research into the characterization of open sets, leading to the development of numerous characterization of open sets.

In 1971, Crossley and Hildebrand [6] introduced the concept of semi-closure of a subset of a topological space. This concept was further explored by Das in 1973 [7], who investigated its applications through utilizing the concept of semi-open sets. Subsequently, in 2008, Navalagi and Gurushantanavar [24] contributed some characterizations for both semi-interior and semi-closure.

Let (X, \mathcal{T}) be a topological space and $G \subseteq X$. Then G is semi-open, if for every $x \in G$ there exists an open set O containing x such that $O \subseteq G \subseteq Cl(O)$. Equivalently, G is semi-open if $G \subseteq Cl(Int(G))$. A subset F of X is semi-closed if its complement $X \setminus F$ is semi-open in X . The semi-closure and semi-interior of G are, respectively, denoted and defined by $sCl(G) = \bigcap \{F : F \text{ is semi-closed and } G \subseteq F\}$ and $sInt(G) = \bigcup \{U : U \text{ is semi-open and } U \subseteq G\}$. Moreover, $sCl(G) = G \cup Int(Cl(G))$.

In 1968, Veličko [32] introduced the notions of θ -continuity, θ -closure, and θ -interior of a subset of a topological space, in order to study the important class of \mathbf{H} -closed spaces in terms

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of arbitrary filterbases. This paper extends Veličko's pioneering work [5, 16, 22, 8, 9], thereby contributing to a deeper understanding of θ -open set.

A subset G of a topological space X is said to be θ -open if for every $x \in G$ there exists an open set O such that $x \in O \subseteq Cl(O) \subseteq G$. The θ -closure and θ -interior of G are, respectively, denoted and defined by $Cl_\theta(G) = \{x \in X : Cl(U) \cap G \neq \emptyset \text{ for every open set } U \text{ containing } x\}$ and $Int_\theta(G) = \{x \in X : Cl(U) \subseteq G \text{ for some open set } U \text{ containing } x\}$. A subset A of X is θ -closed if $Cl_\theta(A) = A$ and θ -open if $Int_\theta(A) = A$. It is known that the collection \mathcal{T}_θ of all θ -open sets forms a topology on X with $\mathcal{T}_\theta \subseteq \mathcal{T}$.

In 1986, Maio and Noiri [21] introduced the concept of semi- θ -open set and provided some characterization and properties. Additionally, the concepts of semi- θ -interior and semi- θ -closure of a subset of a topological space are introduced. In 2022, Singh and Gupta [30] introduced a new class of open set called semi-delta-open sets (briefly δ_s -open sets) and investigated their properties. Moreover, the notion of δ_s -open mappings and δ_s -continuity are introduced and investigated. In 2023, the concept of δ_s -closure, δ_s -interior, δ_s -open mappings and δ_s -continuity was further investigated in [31].

A subset G of a topological space X is δ_s -open [30] (resp., semi- θ -open [21], θ -semi-open [13]) if for every $x \in G$, there exists an open (resp., semi-open) set U containing x such that $Int(sCl(U)) \subseteq G$ (resp., $sCl(U) \subseteq G$, $Cl(U) \subseteq G$). A subset F of X is δ_s -closed (resp., semi- θ -closed, resp., θ -semi-closed) if its complement is $X \setminus F$ is δ_s -open (resp., semi- θ -open, θ -semi-open). A point $p \in X$ is a semi-delta-interior (briefly δ_s -interior) [30] (resp., semi- θ -interior [25]) point of G if there exists a δ_s -open (resp., semi-open) set U in X containing p such that $U \subseteq G$ (resp., $sCl(U) \subseteq G$). We denote by $Int_{\delta_s}(G)$ (resp., $sInt_\theta(G)$) the set of all δ_s -interior (resp., semi- θ -interior) point of G . A point $p \in X$ is a semi-delta-closure (briefly δ_s -closure) [30] (resp., semi- θ -closure [21]) point of G if for every open (resp., semi-open) set U in X containing p , $Int(sCl(U)) \cap G \neq \emptyset$ (resp., $sCl(U) \cap G \neq \emptyset$). The set of all δ_s -closure (resp., semi- θ -closure) point of G is denoted by $Cl_{\delta_s}(G)$ (resp., $sCl_\theta(G)$).

It is known that $sInt(G)$ [24] (resp., $Int_\theta(G)$ [16], $Int_{\delta_s}(G)$ [31], $sInt_\theta(G)$ [4]) is the largest semi-open (resp., θ -open, δ_s -open) set contained in G and $sCl(G)$ [24] (resp., $Cl_\theta(G)$ [16], $Cl_{\delta_s}(G)$ [31], $sCl_\theta(G)$ [21]) is the smallest semi-closed (resp., θ -closed, δ_s -closed, semi- θ -closed) set containing G . Note that $x \in sInt(G)$ [24] (resp., $x \in Int_\theta(G)$ [32]) if and only if there exists a semi-open (resp., open) set U containing x such that $U \subseteq G$ (resp., $Cl(U) \subseteq G$). It is worth noting that $Int(G) \subsetneq sInt(G)$ [24] (resp., $Int_\theta(G) \subseteq Int(G)$ [16], $Int_{\delta_s}(G) \subseteq Int(G)$ [31]) and $sCl(G) \subsetneq Cl(G)$ [24], (resp., $Cl(G) \subseteq Cl_\theta(G)$ [16], $Cl(G) \subseteq Cl_{\delta_s}(G)$ [31], $sCl(G) \subseteq sCl_\theta(G)$ [4]). Moreover, a subset G of X is semi-closed [24] (resp., δ_s -closed [30]) if $sCl(G) = G$ (resp., $Cl_{\delta_s}(G) = G$) and semi-open [24] if $sInt(G) = G$.

Let \mathcal{A} be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let \mathcal{T}_α be the topology on Y_α . The box topology on $\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a basis consisting of all sets of the form $\prod_{\alpha \in \mathcal{A}} U_\alpha$ where U_α is open in Y_α for each $\alpha \in \mathcal{A}$.

In addition, the Tychonoff topology on $\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha)$, where $p_\alpha : \prod_{\alpha \in \mathcal{A}} \{Y_\alpha : \alpha \in \mathcal{A}\} \rightarrow Y_\alpha$, the α th coordinate projection map is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of \mathcal{T}_α , and α ranges over all elements of \mathcal{A} . Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \dots, \alpha_n$ and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}, U_{\alpha_2} \subseteq Y_{\alpha_2}, \dots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \dots \cap \langle U_{\alpha_n} \rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \dots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α , $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, \dots, B_{\alpha_k} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, \dots, k\}$.

Every projection map p_α is a continuous open surjection. It is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in A\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α -th coordinate projection map.

In this paper, we characterize δ_s -continuous functions from an arbitrary topological space into the topological space with either the box or Tychonoff topology. Moreover, we introduce and characterize δ_s -connected spaces and some versions of separation axioms related to δ_s -open sets.

2 Some Properties of δ_s -Open and Semi- θ -Open Sets

In this section, we revisit the concepts of δ_s -open and semi- θ -open sets and determine the connection of these type of open sets to other well-known versions of open sets related to semi-open sets. Moreover, we revisit the concept of δ_s -continuous functions. Throughout, if no confusion arises, let X and Y be topological spaces.

Theorem 2.1. *Let X be a topological space and $A \subseteq X$. Then the following holds:*

- (i) *If A is θ -open, then A is δ_s -open.*
- (ii) *If A is δ_s -open, then A is semi- θ -open.*

Proof. (i) Suppose that A is θ -open and let $x \in A$. Then there exists an open set U containing x such that $Cl(U) \subseteq A$. Note that every closed set is semi-closed set so that $Cl(U)$ is semi-closed. By [24, Theorem 4.21 (i)], $U \subseteq sCl(U) \subseteq Cl(U) \subseteq A$. Also, $Int(sCl(U)) \subseteq Int(Cl(U)) \subseteq Int(A) \subseteq A$. Thus, $Int(sCl(U)) \subseteq A$. Therefore, A is δ_s -open.

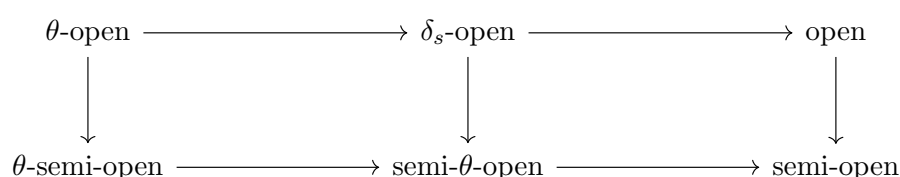
(ii) Assume that A is δ_s -open and let $x \in A$. Then there exists an open set U containing x such that $Int(sCl(U)) \subseteq A$. By [21, Lemma 2.1], $sCl(U) = Int(Cl(U))$ so that $Int(sCl(U)) = Int(Int(Cl(U))) = Int(Cl(U)) \subseteq A$. Since every open set is semi-open set, it follows that there exists a semi-open set U such that $Int(Cl(U)) \subseteq A$. Suppose on the contrary that A is not semi- θ -open. Then for every semi-open set U containing x , $sCl(U) \not\subseteq A$. Since $sCl(U) = U \cup Int(Cl(U))$, $U \cup Int(Cl(U)) \not\subseteq A$. Note that $Int(Cl(U)) \subseteq U \cup Int(Cl(U)) \not\subseteq A$, a contradiction. \square

Corollary 2.2. *Let X be a topological space and $A \subseteq X$. Then the following holds:*

- (i) *If A is θ -closed, then A is δ_s -closed;*
- (ii) *If A is δ_s -closed, then A is semi- θ -closed.*

The following remark is an immediate consequence of [26, Lemma 7.1 and Diagram I], [27, Lemma 6.1], Theorem 2.1, and [31, Remark 19].

Remark 2.3. The following diagram holds for any subset of a topological space.



We note that the above diagram is also true for their respective closed sets. Moreover, none of the reverse implications above is true, as shown in the following examples.

Example 2.4. Let $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Consider the open set $A = \{a, b\}$. Note that $\text{Int}(sCl(\{a\})) = \{a\} \subseteq A$ and $\text{Int}(sCl(\{b\})) = \{b\} \subseteq A$. Hence, A is δ_s -open. To show that A is not θ -open, observe that the only open sets containing a are $\{a\}$, $\{a, b\}$ and X . However, $Cl(X) = Cl(\{a, b\}) = X \not\subseteq \{a, b\}$ and $Cl(\{a\}) = \{a, c\} \not\subseteq \{a, b\}$. Thus, A is not θ -open.

Example 2.5. Let $X = \{a, b, c, d\}$ with topology

$$\mathcal{T} = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}.$$

Consider the open set $B = \{a, c, d\}$. Observe that X and $\{a, c, d\}$ are the only open sets that contain d . However, $\text{Int}(sCl(\{a, c, d\})) = \text{Int}(sCl(X)) = X \not\subseteq B$. Then B is not δ_s -open.

Example 2.6. Consider the set of all real numbers \mathbb{R} with the usual topology. Note that the closed and bounded interval $[1, 3]$, which is not open \mathbb{R} , is semi-open since $[1, 3] \subseteq Cl(\text{Int}([1, 3])) = [1, 3]$. Note also that $Cl([1, 3]) = [1, 3] \subseteq [1, 3]$. This means that $[1, 3]$ is θ -semi-open. This also follows that $[1, 3]$ is semi- θ -open. Since $[1, 3]$ is not open, it also not δ_s -open and not θ -open.

Example 2.7. Consider again the set $A = \{a, b\}$ in Example 2.4, which is a δ_s -open set. It follows that A is semi- θ -open. However, A is not θ -semi-open, since the only semi-open sets containing a are $\{a\}$, $\{a, b\}$, $\{a, c\}$, and X , but $Cl(\{a\}) = Cl(\{a, c\}) = \{a, c\} \not\subseteq A$ and $Cl(\{a, b\}) = Cl(X) = X \not\subseteq A$.

Example 2.8. Let $X = \{a, b\}$ with topology $\mathcal{T} = \{\emptyset, X, \{a\}\}$. Consider the open set $B = \{a\}$, which is also a semi-open set. Note that the only semi-open sets containing a are $\{a\}$ and X , but $sCl(\{a\}) = sCl(X) = X \not\subseteq B$. It follows that B is not a semi- θ -open set.

Remark 2.9. The following statements are true:

- (i) θ -semi-open and δ_s -open sets are independent notions.
- (ii) semi- θ -open and open sets are independent notions.
- (iii) θ -semi-open and open sets are independent notions.

To verify (i), consider again the interval $[1, 3]$ in Example 2.6, which is a θ -semi-open set but not δ_s -open. From Example 2.4, the set $A = \{a, b\}$ is δ_s -open. It has been shown in Example 2.7, that A is not θ -semi-open.

To verify (ii), consider again the interval $[1, 3]$ in Example 2.6, which is a semi- θ -open set but not open. From Example 2.8, the set $B = \{a\}$ is open but not semi- θ -open.

For item (iii), if open implies θ -semi-open, then open implies semi- θ -open, which is a contradiction from (ii). Moreover, every θ -semi-open set is not necessarily open since the interval $[1, 3]$ in Example 2.6 is a θ -semi-open set but not open.

Theorem 2.10. *Let X be a topological space and $A \subseteq X$. Then the following holds:*

- (i) *If A is both open and semi-closed, then A is δ_s -open.*
- (ii) *If A is both semi-open and semi-closed, then A is semi- θ -open.*

Proof. (i) Suppose that A is both open and semi-closed and let $x \in A$. Using [24, Theorem 4.21 (ii)], $A = sCl(A)$. Hence, A is an open set containing x with $Int(sCl(A)) = Int(A) \subseteq A$. Therefore, A is δ_s -open.

(ii) Suppose that A is both semi-open and semi-closed and let $x \in A$. It follows that $A \subseteq sCl(A) = A \subseteq A$. Since A is semi-open and $sCl(A) \subseteq A$, A is semi- θ -open. \square

Theorem 2.11. *Let X be a topological space and $A \subseteq X$. Then the following statements are equivalent:*

- (i) $x \in Int_{\delta_s}(A)$.
- (ii) There exists a δ_s -open set U containing x such that $U \subseteq A$.
- (iii) There exists an open set O containing x such that $Int(sCl(O)) \subseteq A$.

Proof. (i) \Rightarrow (ii) Direct consequence of [30, Definition 3.5].

(ii) \Rightarrow (iii) Suppose that there exists a δ_s -open set U containing x such that $U \subseteq A$. By [30, Definition 3.1], for every $x \in U$ there exists an open set O containing x such that $Int(sCl(O)) \subseteq U \subseteq A$.

(iii) \Rightarrow (i) It follows directly from [30, Definition 3.1 and Remark 4.2 (iii)]. \square

The next results revisit the concept of δ_s -continuous function and investigate its relationship to the other well-known versions of continuity.

Definition 2.12. Let X and Y be topological spaces. A function $f: X \rightarrow Y$ is said to be δ_s -continuous [30] (resp., θ -continuous [16], continuous [10], semi-continuous [18]) on X if $f^{-1}(A)$ is δ_s -open (resp., θ -open, open, semi-open) in X for every open set A in Y .

Theorem 2.13. *Let X and Y be topological spaces and $f_A: X \rightarrow \mathcal{D}$ the characteristic function of subset A of X , where \mathcal{D} is the set $\{0, 1\}$ with the discrete topology. Then f_A is δ_s -continuous if and only if A is both δ_s -open and δ_s -closed.*

Proof. Suppose that f_A is δ_s -continuous on X . Let $O_1 = \{1\}$ and $O_2 = \{2\}$. Then O_1 and O_2 are both open in \mathcal{D} . By assumption, $A = f^{-1}(O_1)$ and $X \setminus A = f^{-1}(O_2)$ are δ_s -open in X . Hence, A is both δ_s -open and δ_s -closed.

Conversely, suppose that A is both δ_s -open and δ_s -closed. Let O be an open set in \mathcal{D} . Then, we have the following

$$f_A^{-1}(O) = \begin{cases} \emptyset & \text{if } O = \emptyset \\ X & \text{if } O = \{0, 1\} \\ A & \text{if } O = \{1\} \\ X \setminus A & \text{if } O = \{0\} \end{cases}$$

Hence, $f_A^{-1}(O)$ is δ_s -open in X . Therefore, f_A is δ_s -continuous on X . \square

Theorem 2.14. *Let X and Y be topological spaces and $f: X \rightarrow Y$ be a function. Then the following holds:*

- (i) If f is θ -continuous on X , then f is δ_s -continuous on X .
- (ii) If f is δ_s -continuous on X , then f is continuous on X .

Proof. (i) Assume that f is θ -continuous on X . Let A be an open set in Y . Then $f^{-1}(A)$ is θ -open in X . Since every θ -open is δ_s -open, it follows that $f^{-1}(A)$ is δ_s -open in X . Hence, f is δ_s -continuous on X .

(ii) Suppose that f is δ_s -continuous on X . Let A be an open set in Y . Then $f^{-1}(A)$ is δ_s -open in X . Since every δ_s -open set is an open set, it follows that $f^{-1}(A)$ is open in X . Therefore, f is continuous on X . \square

It is known that every continuous function is semi-continuous function [18], but not conversely. In view of Theorem 2.14, we have the following remark.

Remark 2.15. The following diagram holds for any subset of a topological space.

$$\theta\text{-continuity} \longrightarrow \delta_s\text{-continuity} \longrightarrow \text{continuity} \longrightarrow \text{semi-continuity}$$

The converse of Remark 2.15 is not necessarily true as shown in the following examples.

Example 2.16. Let $X = \{a, b, c\}$ and $Y = \{1, 2\}$ be two topological spaces with respective topologies $\mathcal{T}_X = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{T}_Y = \{\emptyset, Y, \{2\}\}$. Define $f: X \rightarrow Y$ by $f(a) = f(b) = 2$ and $f(c) = 1$. Clearly, f is δ_s -continuous on X but not θ -continuous on X since $f^{-1}(\{2\}) = \{a, b\}$ is not θ -open in X .

Example 2.17. Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3\}$ be topological spaces with respective topologies $\mathcal{T}_X = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}$ and $\mathcal{T}_Y = \{\emptyset, Y, \{2, 3\}\}$. Define $f: X \rightarrow Y$ by $f(a) = f(d) = 2$, $f(b) = 3$ and $f(c) = 1$. Note that f is continuous on X but not δ_s -continuous on X .

Example 2.18. Let $X = \{x, y, z, w\}$ and $Y = \{a, b, c, d\}$ be topological spaces with respective topologies $\mathcal{T}_X = \{\emptyset, X, \{y\}, \{z\}, \{y, z\}, \{x, z, w\}\}$ and $\mathcal{T}_Y = \{\emptyset, Y, \{b\}, \{a, d\}, \{a, b, d\}\}$. Define $f: X \rightarrow Y$ by $f(x) = c$, $f(y) = b$, $f(z) = a$ and $f(w) = d$. Then f is semi-continuous on X but not continuous on X .

3 δ_s -Continuous Functions in the Box Topology

In this section, we provide a characterization of δ_s -continuous functions from an arbitrary topological space into the topological space with the box topology.

Theorem 3.1. Let X be a topological space and $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a topological space with either the box or Tychonoff topology. A function $f: X \rightarrow Y$ is δ_s -continuous on X if and only if $p_\alpha \circ f$ is δ_s -continuous on X for every $\alpha \in \mathcal{A}$.

Proof. Assume that f is δ_s -continuous on X . Let $\alpha \in \mathcal{A}$ and O_α be open in Y_α . Since p_α is continuous, $p_\alpha^{-1}(O_\alpha)$ is open in Y . Hence, $f^{-1}(p_\alpha^{-1}(O_\alpha)) = (p_\alpha \circ f)^{-1}(O_\alpha)$ is δ_s -open in X . Thus, $p_\alpha \circ f$ is δ_s -continuous for every $\alpha \in \mathcal{A}$.

Conversely, suppose that each coordinate function $p_\alpha \circ f$ is δ_s -continuous on X for every $\alpha \in \mathcal{A}$. Let B_α be open in Y_α for each $\alpha \in \mathcal{A}$. Then $\langle B_\alpha \rangle$ is a subbasic open set in Y and $(p_\alpha \circ f)^{-1}(B_\alpha) = f^{-1}(p_\alpha^{-1}(B_\alpha)) = f^{-1}(\langle B_\alpha \rangle)$ is δ_s -open in X . Therefore, f is δ_s -continuous on X . \square

Corollary 3.2. Let X be a topological space and $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a topological space with either the box or Tychonoff topology and $f_\alpha : X \rightarrow Y_\alpha$ be a function for each $\alpha \in \mathcal{A}$. Let $f: X \rightarrow Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then f is δ_s -continuous on X if and only if each f_α is δ_s -continuous on X for each $\alpha \in \mathcal{A}$.

Before we proceed with results involving closure and interior operators with respect to δ_s -open sets in the box topology, we shall consider first the following results related to semi-open sets.

Theorem 3.3. *Let $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a topological space with the box topology and $\emptyset \neq O_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. Then each O_α is semi-open in Y_α if and only if $O = \prod\{O_\alpha : \alpha \in \mathcal{A}\}$ is semi-open in Y .*

Proof. Following the proof of [12, Theorem 3.4], $O = \prod\{O_\alpha : \alpha \in \mathcal{A}\}$ is semi-open in Y if each O_α is semi-open in Y_α .

Conversely, suppose that $\prod\{O_\alpha : \alpha \in \mathcal{A}\}$ is semi-open in Y . Then there exists a basic open set $G = \prod\{G_\alpha : \alpha \in \mathcal{A}\}$ in Y such that $G \subseteq O \subseteq Cl(G) = \prod\{Cl(G_\alpha : \alpha \in \mathcal{A})\}$. Hence, for all $\alpha \in \mathcal{A}$, G_α is open in Y_α and $G_\alpha \subseteq O_\alpha \subseteq Cl(G_\alpha)$. Thus, each O_α is semi-open in Y_α . \square

The proof of the following theorem is similar to [12, Theorem 3.5], hence omitted.

Theorem 3.4. *Let $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a topological space with the box topology. If $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, then $sCl(\prod\{A_\alpha : \alpha \in \mathcal{A}\}) \subseteq \prod\{sCl(A_\alpha) : \alpha \in \mathcal{A}\}$.*

Since the interior operator is preserved in the box topology, the following result holds.

Theorem 3.5. *Let $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a topological space with the box topology. If $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, then $Cl_{\delta_s}(\prod\{A_\alpha : \alpha \in \mathcal{A}\}) \subseteq \prod\{Cl_{\delta_s}(A_\alpha) : \alpha \in \mathcal{A}\}$.*

Proof. Assume that $x = \langle x_\alpha \rangle \in Cl_{\delta_s}(\prod\{A_\alpha : \alpha \in \mathcal{A}\})$. Suppose on the contrary that $x \notin \prod\{Cl_{\delta_s}(A_\alpha) : \alpha \in \mathcal{A}\}$. Then for some $\beta \in \mathcal{A}$, $x_\beta \notin Cl_{\delta_s}(A_\beta)$. This means that there exists an open set V_β containing x_β such that $Int(sCl(V_\beta)) \cap A_\beta = \emptyset$. Let $V_\beta \times \prod_{\alpha \neq \beta} Y_\alpha =: \langle V_\beta \rangle$. Then $\langle V_\beta \rangle$ is a basic open in Y containing $\langle x_\alpha \rangle$. By Theorem 3.4, $sCl(\langle V_\beta \rangle) \subseteq \langle sCl(V_\beta) \rangle$. It follows that $Int(sCl(\langle V_\beta \rangle)) \subseteq Int(\langle sCl(V_\beta) \rangle) = \langle Int(sCl(V_\beta)) \rangle$. Thus, we have

$$\begin{aligned} Int(sCl(\langle V_\beta \rangle)) \cap \prod\{A_\alpha : \alpha \in \mathcal{A}\} &\subseteq \langle Int(sCl(V_\beta)) \rangle \cap \prod\{A_\alpha : \alpha \in \mathcal{A}\} \\ &= (Int(sCl(V_\beta)) \cap A_\beta) \times (\prod\{Y_\alpha : \alpha \neq \beta\} \cap \prod\{A_\alpha : \alpha \neq \beta\}) \\ &= (Int(sCl(V_\beta)) \cap A_\beta) \times \prod\{Y_\alpha \cap A_\alpha : \alpha \neq \beta\} \\ &= (Int(sCl(V_\beta)) \cap A_\beta) \times \prod\{A_\alpha : \alpha \neq \beta\} \\ &= \emptyset, \end{aligned}$$

a contradiction. Therefore, $x \in \prod\{Cl_{\delta_s}(A_\alpha) : \alpha \in \mathcal{A}\}$. \square

Theorem 3.6. *Let $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a topological space with the box topology. If $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, then $\prod\{Int_{\delta_s}(A_\alpha) : \alpha \in \mathcal{A}\} \subseteq Int_{\delta_s}(\prod\{A_\alpha : \alpha \in \mathcal{A}\})$.*

Proof. Let $x = \langle x_\alpha \rangle \in \prod\{Int_{\delta_s}(A_\alpha) : \alpha \in \mathcal{A}\}$. Then $x_\alpha \in Int_{\delta_s}(A_\alpha)$ for all $\alpha \in \mathcal{A}$. This means that for each $\alpha \in \mathcal{A}$, there exists an open set V_α containing x_α such that $Int(sCl(V_\alpha)) \subseteq A_\alpha$. Note that $\prod\{V_\alpha : \alpha \in \mathcal{A}\}$ is open containing x . Then by Theorem 3.4, $sCl(\prod\{V_\alpha : \alpha \in \mathcal{A}\}) \subseteq \prod\{sCl(V_\alpha) : \alpha \in \mathcal{A}\}$. Thus, we have

$$\begin{aligned} Int(sCl(\prod\{V_\alpha : \alpha \in \mathcal{A}\})) &\subseteq Int(\prod\{sCl(V_\alpha) : \alpha \in \mathcal{A}\}) \\ &= \prod\{Int(sCl(V_\alpha)) : \alpha \in \mathcal{A}\} \\ &\subseteq \prod\{A_\alpha : \alpha \in \mathcal{A}\}. \end{aligned}$$

Therefore, $x \in Int_{\delta_s}(\prod\{A_\alpha : \alpha \in \mathcal{A}\})$. \square



It is worth noting that the interior operator is preserved in the product space provided that the collection is finite. Following the same argument as in Theorem 3.5 and Theorem 3.6, the following result holds.

Corollary 3.7. *Let $Y = \prod\{Y_i: 1 \leq i \leq n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i \in \{1, 2, \dots, n\}$. Then $Cl_{\delta_s}(\langle A_1, \dots, A_n \rangle) \subseteq \langle Cl_{\delta_s}(A_1), \dots, Cl_{\delta_s}(A_n) \rangle$.*

Corollary 3.8. *Let $Y = \prod\{Y_i: 1 \leq i \leq n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i \in \{1, 2, \dots, n\}$. Then $\langle Int_{\delta_s}(A_1), \dots, Int_{\delta_s}(A_n) \rangle \subseteq Int_{\delta_s}(\langle A_1, \dots, A_n \rangle)$.*

Following the same argument as in Theorem 3.3, we have the following result.

Theorem 3.9. *Let $Y = \prod\{Y_\alpha: \alpha \in \mathcal{A}\}$ be a topological space with the box topology and $\emptyset \neq O_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. If for each $\alpha \in \mathcal{A}$, O_α is δ_s -open in Y_α , then $O = \prod\{O_\alpha: \alpha \in \mathcal{A}\}$ is δ_s -open in Y .*

Corollary 3.10. *Let $Y = \prod\{Y_i: 1 \leq i \leq n\}$ be a finite product space and $\emptyset \neq O_i \subseteq Y_i$ for each $i \in \{1, 2, \dots, n\}$. If each O_i is δ_s -open in Y_i , then $O = \langle O_1, \dots, O_n \rangle$ is δ_s -open in Y .*

Theorem 3.11. *Let $X = \prod\{X_\alpha: \alpha \in \mathcal{A}\}$ and $Y = \prod\{Y_\alpha: \alpha \in \mathcal{A}\}$ be topological spaces with the box topology and for $\alpha \in \mathcal{A}$ and let $f_\alpha: X_\alpha \rightarrow Y_\alpha$ be a function. If each f_α is δ_s -continuous on X_α , then the function $f: X \rightarrow Y$ defined by $f(\langle x_\alpha \rangle) = \langle f_\alpha(x_\alpha) \rangle$ is δ_s -continuous on X .*

Proof. Let $\prod_{\alpha \in \mathcal{A}} V_\alpha$ be a basic open set in Y . Then $f^{-1}(\prod_{\alpha \in \mathcal{A}} V_\alpha) = \prod_{\alpha \in \mathcal{A}} f_\alpha^{-1}(V_\alpha)$. Since each f_α is δ_s -continuous, $f_\alpha^{-1}(V_\alpha)$ is δ_s -open in X_α . Let $x = \langle x_\alpha \rangle \in f^{-1}(\prod_{\alpha \in \mathcal{A}} V_\alpha)$. Then $x_\alpha \in f_\alpha^{-1}(V_\alpha)$ for all $\alpha \in \mathcal{A}$. This means that there exists an open set O_α containing x_α such that $Int(sCl(O_\alpha)) \subseteq f_\alpha^{-1}(V_\alpha)$. Note that $\prod\{O_\alpha: \alpha \in \mathcal{A}\}$ is open in X containing x and by Theorem 3.4, $Int(sCl(\prod_{\alpha \in \mathcal{A}} O_\alpha)) \subseteq \prod_{\alpha \in \mathcal{A}} Int(sCl(O_\alpha)) \subseteq \prod_{\alpha \in \mathcal{A}} f_\alpha^{-1}(V_\alpha) = f^{-1}(\prod_{\alpha \in \mathcal{A}} V_\alpha)$. This implies that $f^{-1}(\prod_{\alpha \in \mathcal{A}} V_\alpha)$ is δ_s -open in X . Therefore, f is δ_s -continuous on X . \square

4 δ_s -Connected Space and Some Version of Separation Axioms

In this section, we introduce and characterize δ_s -connected space. Moreover, some versions of separation axioms related to δ_s -open sets are introduced and characterized.

Definition 4.1. A topological space X is said to be a δ_s -connected if it is not a union of two nonempty disjoint δ_s -open sets. Otherwise, X is δ_s -disconnected. A subset B of X is δ_s -connected if it is δ_s -connected as a subspace of X .

Theorem 4.2. *Let X be a topological space. Then the following statements are equivalent:*

- (i) X is δ_s -connected;
- (ii) The only subsets of X that are both δ_s -open and δ_s -closed are \emptyset and X ;
- (iii) No δ_s -continuous function $f: X \rightarrow \mathcal{D}$ is surjective.

Proof. (i) \Rightarrow (ii) Assume that X is δ_s -connected and let $A \subseteq X$. Let A be both δ_s -open and δ_s -closed. Then $X \setminus A$ is both δ_s -open and δ_s -closed. Note that $X = A \cup (X \setminus A)$. Since X is δ_s -connected, it follows that A is either \emptyset or X .

(ii) \Rightarrow (iii) Suppose that \emptyset and X are the only subsets of X that are both δ_s -open and δ_s -closed. Let $f: X \rightarrow \mathcal{D}$ be a δ_s -continuous surjection. Then $f^{-1}(\{0\}) \neq \emptyset$ and $f^{-1}(\{0\}) \neq X$. Note that every set in \mathcal{D} is open. Since $\{0\}$ is both open and closed in \mathcal{D} , it follows that $f^{-1}(\{0\})$ is both δ_s -open and δ_s -closed in X , a contradiction.

(iii) \Rightarrow (i) Suppose that no δ_s -continuous function $f : X \rightarrow \mathcal{D}$ is surjective. Let $X = A \cup B$, where A and B are disjoint nonempty δ_s -open sets. Then A and B are also δ_s -closed sets. Consider the characteristic function $f_A : X \rightarrow \mathcal{D}$ of $A \subseteq X$. By Theorem 2.13, f_A is δ_s -continuous, a contradiction. Thus, X is δ_s -connected. \square

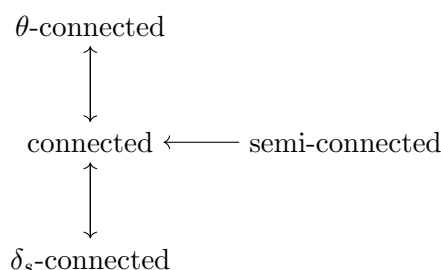
Theorem 4.3. *Let X be a topological space. Then the following holds:*

- (i) X is connected if and only if X is θ -connected [16];
- (ii) If X is semi-connected, then X is connected [18].

In view of Theorem 2.1 and Theorem 4.3 (i), the proof of the following theorem is omitted.

Theorem 4.4. *Let X be a topological space. Then X is δ_s -connected if and only if X is connected.*

Remark 4.5. The following diagram holds for any subset of a topological space.



The reverse implication for connected and semi-connected spaces is not true, as shown in the following example.

Example 4.6. Consider again the topological space in Example 2.6. Then X is connected but not semi-connected.

The following results and versions of separation axioms are related to δ_s -open sets.

Definition 4.7. A topological space X is said to be

- (i) *semi-delta-Hausdorff* (briefly δ_s -Hausdorff) if given any pair of distinct points $p, q \in X$, there exist disjoint δ_s -open sets U and V such that $p \in U$ and $q \in V$;
- (ii) *semi-delta-regular* (briefly δ_s -regular) if for each closed set F and each point $p \notin F$, there exist disjoint δ_s -open sets U and V such that $p \in U$ and $F \subseteq V$;
- (iii) *semi-delta-normal* (briefly δ_s -normal) if for every pair of disjoint closed sets E and F of X , there exist disjoint δ_s -open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Theorem 4.8. *Let X be a topological space. Then the following statements are equivalent:*

- (i) X is δ_s -Hausdorff;
- (ii) Let $x \in X$. For $y \neq x$, there exists a δ_s -open set U containing x such that $y \notin Cl_{\delta_s}(U)$;
- (iii) For each $x \in X$, $V = \bigcap \{Cl_{\delta_s}(U) : U \text{ is } \delta_s\text{-open set containing } x\} = \{x\}$.

Proof. (i) \Rightarrow (ii): Assume that (i) holds. Then for every distinct points $x, y \in X$, there exist distinct δ_s -open sets U and W such that $x \in U$ and $y \in W$. Thus, $y \notin Cl_{\delta_s}(U)$.

(ii) \Rightarrow (iii): Assume that (ii) holds. It is not difficult to see that $x \in V$. By assumption, for every $x \neq y$, there exists a δ_s -open set U containing x such that $y \notin Cl_{\delta_s}(U)$. This means that $y \notin V$. Since y is arbitrary, it follows that $V = \{x\}$.

(iii) \Rightarrow (i): Assume that (iii) holds. Let $x, y \in X$ with $x \neq y$. By assumption, there exists a δ_s -open set W containing y such that $U \cap W = \emptyset$. Therefore, X is δ_s -Hausdorff. \square

Theorem 4.9. *Let X be a topological space. Then the following statements are equivalent:*

- (i) X is δ_s -regular;
- (ii) For each $x \in X$ and an open set U containing x , there exists a δ_s -open set V such that $x \in V \subseteq Cl_{\delta_s}(V) \subseteq U$;
- (iii) For each $x \in X$ and closed set F with $x \notin F$, there exists a δ_s -open set V containing x such that $F \cap Cl_{\delta_s}(V) = \emptyset$.

Proof. (i) \Rightarrow (ii): Assume that X is δ_s -regular. Let $x \in X$ and U be an open set in X containing x . Then $X \setminus U$ is closed in X with $x \notin X \setminus U$. By assumption, there exist disjoint δ_s -open sets V and W such that $x \in V$ and $X \setminus U \subseteq W$. Note that $V \cap W = \emptyset$ so that $V \subseteq X \setminus W$. Then, we have $Cl_{\delta_s}(V) \subseteq Cl_{\delta_s}(X \setminus W) = X \setminus Int_{\delta_s}(W) = X \setminus W$, since W is δ_s -open. Hence, $Cl_{\delta_s}(V) \cap W = \emptyset$. Thus, $Cl_{\delta_s}(V) \cap (X \setminus U) \subseteq Cl_{\delta_s}(V) \cap W = \emptyset$. It follows that $Cl_{\delta_s}(V) \subseteq U$. Therefore, $x \in V \subseteq Cl_{\delta_s}(V) \subseteq U$.

(ii) \Rightarrow (iii): Assume that (ii) holds. Let $x \in X$ and F be a closed set in X with $x \notin F$. Then $X \setminus F$ is an open set in X with $x \in X \setminus F$. By assumption, there exists a δ_s -open set V containing x such that $V \subseteq Cl_{\delta_s}(V) \subseteq X \setminus F$. Hence, $F \cap Cl_{\delta_s}(V) = \emptyset$.

(iii) \Rightarrow (i): Assume that (iii) holds. Let $x \in X$ and F be a closed set in X with $x \notin F$. By assumption, there exists a δ_s -open set V containing x such that $F \cap Cl_{\delta_s}(V) = \emptyset$. Since $Cl_{\delta_s}(V)$ is δ_s -closed, it follows that $X \setminus Cl_{\delta_s}(V)$ is δ_s -open. Also, $F \subseteq X \setminus Cl_{\delta_s}(V)$. Since $Cl_{\delta_s}(V)$ is the smallest δ_s -closed set containing V , that is, $V \subseteq Cl_{\delta_s}(V)$, it follows that $V \cap X \setminus Cl_{\delta_s}(V) = \emptyset$. Therefore, X is δ_s -regular. \square

Theorem 4.10. *Let X be a topological space. Then the following statements are equivalent:*

- (i) X is δ_s -normal;
- (ii) For each closed set A and an open set U containing A , there exists a δ_s -open set V containing A such that $Cl_{\delta_s}(V) \subseteq U$;
- (iii) For each pair of disjoint closed sets A and B , there exists a δ_s -open set V containing A such that $Cl_{\delta_s}(V) \cap B = \emptyset$.

Proof. (i) \Rightarrow (ii): Assume that X is δ_s -normal. Let A be a closed set in X and U be an open set in X containing A . Then A and $X \setminus U$ are disjoint closed sets in X . By assumption, there exist disjoint δ_s -open sets V and O such that $A \subseteq V$ and $X \setminus U \subseteq O$. Since $V \cap O = \emptyset$ and $X \setminus U \subseteq O$, it follows that $X \setminus O \subseteq U$ and $V \subseteq X \setminus O$. Thus, we have

$$Cl_{\delta_s}(V) \subseteq Cl_{\delta_s}(X \setminus O) = X \setminus Int_{\delta_s}(O) = X \setminus O,$$

since O is δ_s -open. Thus, $Cl_{\delta_s}(V) \subseteq U$.

(ii) \Rightarrow (iii): Assume that (ii) holds. Let A and B be disjoint closed sets in X . Then $X \setminus B$ is open in X with $A \subseteq X \setminus B$. By assumption, there exists a δ_s -open set V containing A such that $Cl_{\delta_s}(V) \subseteq X \setminus B$. Hence, $Cl_{\delta_s}(V) \cap B = \emptyset$.

(iii) \Rightarrow (i): Assume that (iii) holds. Let A and B be disjoint closed sets in X . Then there exists δ_s -open set V containing A such that $Cl_{\delta_s}(V) \cap B = \emptyset$. It follows that $B \subseteq X \setminus Cl_{\delta_s}(V)$. Since $Cl_{\delta_s}(V)$ is δ_s -closed, it follows that $X \setminus Cl_{\delta_s}(V)$ is δ_s -open. Also, $A \subseteq V$ and $B \subseteq X \setminus Cl_{\delta_s}(V)$. Since $V \subseteq Cl_{\delta_s}(V)$, $V \cap X \setminus Cl_{\delta_s}(V) = \emptyset$. Therefore, X is δ_s -normal. \square

A topological space X is said to be a T_1 -space if for each $p, q \in X$ with $p \neq q$, there exist open sets U and V such that $p \in U$, $q \notin U$ and $q \in V$, $p \notin V$.

Theorem 4.11. *Let X be a T_1 -space. Then the following statements hold*

- (i) *If X is δ_s -normal, then X is δ_s -regular;*
- (ii) *If X is δ_s -regular, then X is δ_s -Hausdorff.*

Proof. (i) Assume that X is δ_s -normal. Let F be a closed set and $x \notin F$. Since X is a T_1 -space, $\{x\}$ is closed in X . By assumption, there exist disjoint δ_s -open sets U and V such that $\{x\} \subseteq U$ and $F \subseteq V$. Note that $x \in \{x\}$ so that $x \in U$. Hence, X is δ_s -regular.

(ii) Assume that X is δ_s -regular. Let $x, y \in X$ with $x \neq y$. Then there exist open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. It follows that $X \setminus U$ is closed with $x \notin X \setminus U$ and $y \in X \setminus U$. By assumption, there exist disjoint δ_s -open sets A and B such that $x \in A$ and $X \setminus U \subseteq B$. Clearly, $y \in B$. Therefore, X is δ_s -Hausdorff. \square

Summary and Recommendations

This paper extends the concept of δ_s -continuous functions from an arbitrary topological space to a topological space with either the box or Tychonoff topology. In addition, this paper introduces the concepts of δ_s -connected space, δ_s -Hausdorff space, δ_s -regular space, and δ_s -normal space. Future research may focus on developing the concept of δ_s -compact space and establishing versions of Urysohn's Lemma and the Tietze Extension Theorem for δ_s -open sets.

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