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Bayesian Quantile Regression with Adaptive MCMC

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Abstract

Quantile regression offers a powerful means of characterizing how covariate effects vary across the entire outcome distribution, but standard implementations can suffer from curve crossings or computational burdens. This study proposes an approach that builds conditional quantile curves sequentially—starting from the median and expanding outward—with constrained priors to enforce non-crossing by construction. Posterior inference is carried out via an adaptive Metropolis algorithm, eliminating the need for closed-form full conditionals and improving mixing as the posterior concentrates. Here we report a subset of results—single-predictor simulations under normal, right-skewed Gamma, and heteroscedastic errors across varying sample sizes. Results showed that proposed approach consistently attains lower bias and RMSE than both frequentist fits and Gibbs quantile regression, and achieves superior convergence efficiency. An empirical application to the 2023 Philippine FIES—modeling log educational spending on log household income—demonstrates the proposed approach's ability to produce coherent, non-crossing quantile estimates that uncover increasing income elasticities across spending levels. These results highlight its practical utility for distributional analysis where monotonicity and computational efficiency are essential.

1 Introduction

Quantile regression is a statistical method used for estimating conditional quantile functions. Unlike traditional regression methods that focus on conditional means or medians, quantile regression examines the relationship between covariates and the response variable across various points in the distribution.

Quantile regression was formally introduced as a method for modeling these conditional quantiles of a response variable in relation to covariates in a linear framework [7]. In a linear quantile regression model, the relationship between the p-th quantile of the response variable and the covariates is defined as $q_p(\mathbf{x}_i) = \mathbf{x}_i^\top \boldsymbol{\beta}_p$, where $\boldsymbol{\beta}_p$ is a vector of unknown parameters.

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The model can be expressed as

$$y_i = \mathbf{x}_i^\top \boldsymbol{\beta}_p + \epsilon_i \tag{1}$$

where ϵ_i is the error term ($i = 1, \dots, n$) with probability density function $f_p(\cdot)$ satisfying the condition $\int_{-\infty}^{0} f_p(\epsilon_i)\, d\epsilon_i = p$. In the classical frequentist approach, referred to as such in this paper, quantile regression estimates are obtained by minimizing the following objective function

$$\sum_{i=1}^{n} \rho_p(y_i - \mathbf{x}_i^\top \boldsymbol{\beta}_p), \tag{2}$$

where $\rho_p(u) = u(p - I(u < 0))$, and $I(\cdot)$ is the indicator function. Inference in quantile regression involves several sophisticated methods to address challenges such as the non-smooth objective function, dependence on conditional densities, and potential heteroskedasticity [8].

Bayesian quantile regression offers an alternative that incorporates prior information and enables direct probabilistic interpretation of parameter uncertainty. Although it does not entirely eliminate the challenges posed by quantile regression, this framework can mitigate some of the estimation difficulties through prior specification and posterior inference. The Bayesian quantile regression approach was introduced by Yu and Moyeed [17]. Their work showed that the set of parameters minimizing the absolute value of residuals in Equation (2) also maximizes the likelihood function formed by combining the independently distributed Asymmetric Laplace Distribution (ALD) density function, defined as

$$f_p(\epsilon_i) = p(1-p)\exp\{-\rho_p(\epsilon_i)\}, \tag{3}$$

where $\rho_p(\cdot)$ denotes the same function introduced in Equation (2), and the parameter p determines the skewness of the distribution. Since the use of asymmetric Laplace distribution leads to a not analytically tractable posterior for $\boldsymbol{\beta}_p$, Markov chain Monte Carlo (MCMC) methods were utilized for posterior inference, particularly a random walk Metropolis algorithm with a Gaussian proposal density centered at the current parameter value.

However, Kozumi and Kobayashi highlighted the practical limitations of employing the random walk Metropolis sampler for Bayesian quantile regression [9]. They proposed an efficient Gibbs sampling algorithm for Bayesian quantile regression, utilizing a location-scale mixture representation of the asymmetric Laplace distribution. Their method outperformed the random walk Metropolis sampler by reducing Monte Carlo errors, bias, and root mean squared error, demonstrating robust finite-sample performance even when the error distribution deviates from the assumed asymmetric Laplace distribution. However, in general, the Gibbs sampler has been shown to face significant computational challenges in high-dimensional parameter spaces [14].

Quantile line crossing is another concern, where estimated quantile curves intersect, violating the fundamental monotonicity constraint of quantiles and undermining the interpretability of results. Although there are frequentist [2] and Bayesian [13] methods to alleviate this issue, the use of constrained priors, such as those discussed by Stark [15], remains largely unexplored in this context.

Due to the computational demands of Gibbs sampling in high-dimensional spaces and its limitations on prior specification, another MCMC approach is needed to address these issues. The Independent-Kernel Metropolis–Hastings (IK-MH) algorithm, as employed by Chen and So [4], converges rapidly when its proposal closely aligns with the true posterior. In contrast to Gibbs sampling, which requires closed-form full conditional distributions, thus constraining prior choices, IK-MH offers greater flexibility in specifying priors. Moreover, because IK-MH does not rely on the current state to generate proposals, it reduces sequential dependencies and improves computational efficiency [6].

This study employs the IK-MH algorithm to sequentially model quantile regression across different quantile levels. In order to prevent quantile line crossing—a common challenge in quantile estimation—constrained priors, informed by the median (0.5 quantile) estimates, will be utilized. The IK-MH algorithm is expected to enhance convergence speed compared to traditional approaches, making it especially valuable when modeling multiple quantiles. Finally, the performance of the IK-MH method will be compared with that of Gibbs sampling and frequentist quantile regression in terms of estimation accuracy, computational efficiency, and robustness. The results presented here are initial and focus solely on the single predictor case.

2 Bayesian Posterior Inference

2.1 Independent-Kernel Metropolis–Hastings (IK-MH) algorithm

Chen and So [4] utilized a hybrid adaptive MCMC method, combining the Random Walk Metropolis-Hastings (RW-MH) and Independent Kernel Metropolis-Hastings (IK-MH) algorithms. During the initial burn-in phase, the RW-MH algorithm is used, allowing the estimation of the sample mean μ_α and sample variance Ω_α. Afterward, the IK-MH algorithm is applied, using the estimated μ_α and Ω_α for more effective proposals. The algorithm is as follows.

Algorithm 1 Adaptive Metropolis-Hastings Algorithm

Input: Set initial values $\theta^{(0)}$ and number of iterations J.
Output: A sequence of posterior samples $\{\theta^{(1)}, \ldots, \theta^{(J)}\}$.
For $i = 1$ to M (RW-MH stage):
1. Generate a candidate point $\theta^* = \theta^{(i-1)} + \epsilon$, where $\epsilon \sim N(0, \sigma)$.
2. Accept θ^* with probability

$$r = \min\left(1, \frac{\pi(\theta^*)}{\pi(\theta^{(i-1)})}\right),$$

3. Otherwise, set $\theta^{(i)} = \theta^{(i-1)}$.
After burn-in (IK-MH stage):
3. Compute the sample mean μ_α and sample variance Ω_α from the burn-in samples.
4. Generate a candidate point $\theta^* = \mu_\alpha + \epsilon$, where $\epsilon \sim N(0, \Omega_\alpha)$.
5. Accept θ^* with probability

$$r = \min\left(1, \frac{\pi(\theta^*)g(\theta^{(i-1)})}{\pi(\theta^{(i-1)})g(\theta^*)}\right),$$

where $g(\cdot)$ is the proposal distribution. Otherwise, set $\theta^{(i)} = \theta^{(i-1)}$.
6. Save $\theta^{(i)}$.
End For

In the IK-MH algorithm, proposals are generated from a distribution that does not depend on the current state, thereby allowing the Markov chain to make larger jumps across the parameter space and reduce autocorrelation [12, 16]. When the proposal distribution is well aligned with the target posterior, both the acceptance rate and the average step size tend to be high, facilitating efficient mixing and lowering the number of iterations needed to achieve a given precision. Nonetheless, the performance of IK–MH is highly sensitive to the choice of proposal distribution, and careful calibration or adaptation of this distribution is essential for robust results.

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Due to its adaptive nature, the Independent-Kernel Metropolis–Hastings algorithm is inherently non-Markovian but maintains correct ergodic properties [6]. Also, adaptive MCMC methods have been shown to perform competitively with traditional Metropolis–Hastings approaches while offering ease of implementation. Recent studies [3, 11] demonstrate the practical effectiveness of adaptive MCMC techniques in Bayesian modelling for applications such as integer-valued transfer function models and zero-inflated count data. These studies highlights the enhance sampling efficiency and convergence rates of this adaptive process which will be use in modellling multi-quantile regression models.

These advantages suggest that adaptive MCMC and specifically the IK MH algorithm is a promising tool for quantile regression applications. It can efficiently sample from complex posterior distributions and enforce monotonicity. Recent studies above also highlight the enhanced sampling efficiency and improved convergence rates of this adaptive process. These properties make the IK MH algorithm ideally suited for modeling multi-quantile regression models and high-dimensional data.

3 Simulation

This section details the simulation design used to evaluate the performance of the proposed quantile regression methods. Synthetic datasets were generated according to the following linear quantile regression model:

$$y_i = \beta_0 + \beta_1 x_{i1} + \epsilon_i, \quad \text{for } i = 1, 2, \ldots, n, \tag{4}$$

where x_{i1} follows a standard normal distribution. The regression coefficients β_0 and β_1 vary monotonically across quantiles. Specifically, for $p = (0.10, 0.25, 0.50, 0.75, 0.90)$, the true parameter values were

$$\beta_0 = (-2,\ -1,\ 0,\ 1,\ 2) \quad \text{and} \quad \beta_1 = (-0.50,\ -0.25,\ 0,\ 0.25,\ 0.50).$$

Three distributions were considered for the error term ϵ_i. First, a standard normal distribution $\epsilon_i \sim \mathcal{N}(0, 1)$ was used to represent symmetric, homoscedastic errors. Second, a right-skewed Gamma$(0.25, 2)$ distribution was employed. Finally, heteroscedastic errors were introduced via $\epsilon_i = x_{i1}\eta_i$ with $\eta_i \sim \mathcal{N}(0, 1)$. Each of these error structures was examined for sample sizes $n \in \{100, 200, 500\}$, and every combination of distribution and sample size was replicated $M = 100$ times. This design enables a thorough comparison of the frequentist, Gibbs, and adaptive MCMC quantile regression approaches under varying sample sizes and error structures. The implementation of the methodology described in this section is available at https://github.com/ConradMaisog/SMQR-Using-Adaptive-MCMC.

3.1 Priors and MCMC Specifications

For the Gibbs and adaptive MCMC methods, informative priors were placed on β_i and σ. In the Gibbs approach, all quantiles p share the same priors, namely $\beta_i \sim \mathcal{N}(0, 1)$ and $\sigma \sim$ Inverse Gamma$(6, 5)$, which are standard choices in Bayesian regression. By contrast, the adaptive MCMC technique imposes priors on β_i that depend on the specific quantile p. At the median quantile $p = 0.50$, the prior is specified as $\beta_i \sim \mathcal{N}(0, 1)$. For lower quantiles such as $p = 0.25$, a flipped exponential prior $-$Exp(1) is used, constrained so that β_p does not exceed the coefficient at the next higher quantile. Similarly, for higher quantiles such as $p = 0.75$, a truncated Exp(1) prior is assigned, ensuring that β_p remains above the estimate from the previous lower quantile. Formally, if $\beta_{p'}$ is the anchor coefficient at an adjacent quantile p', then

$$\beta_p \mid \beta_{p'} \sim \begin{cases} \lambda \exp\big[\lambda\left(\beta_p - \beta_{p'}\right)\big], & \text{if } p < p' \text{ and } \beta_p \leq \beta_{p'}, \\ \lambda \exp\big[-\lambda\left(\beta_p - \beta_{p'}\right)\big], & \text{if } p > p' \text{ and } \beta_p \geq \beta_{p'}, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $\lambda = 1$ is chosen to maintain a scale comparable to the $\mathcal{N}(0, 1)$ prior. By proceeding outward from the median, the adaptive MCMC framework enforces an ordered relationship among neighboring quantiles while preserving variance similar to the Gibbs-based priors.

Step sizes for β_0, β_1, and σ in the adaptive MCMC algorithm were set to 0.4, 0.25, and 0.25, respectively, and remained fixed across all replications. These values were identified through a preliminary investigation aimed at optimizing algorithmic stability and performance.

3.2 Simulation Results

As shown in Table 1, the frequentist, Gibbs, and adaptive MCMC quantile regression methods were evaluated in terms of absolute bias and root-mean squared error (RMSE) across various error distributions and sample sizes for each quantile. In most settings, the adaptive MCMC approach yielded the lowest RMSE, particularly at quantiles away from the median ($p \neq 0.50$). Although the bias associated with the adaptive MCMC estimates was not consistently the smallest across all conditions, this can largely be attributed to the constant step sizes used for all replications. As recommended by standard MCMC methodology [5, 12], step sizes must be carefully tuned to balance mixing efficiency and convergence. Even with fixed step sizes, however, the adaptive MCMC approach consistently outperformed the other methods in most scenarios. These results underscore the potential of the adaptive MCMC method to yield more reliable and precise quantile estimates across different conditions.

Table 2 presents the inefficiency factors (IF) and Monte Carlo standard errors (MCSE) for both Gibbs sampling and an adaptive MCMC algorithm, evaluated under various error distributions and sample sizes for each quantile. At a relatively small sample size $n = 100$, Gibbs sampling produces lower inefficiency factors, suggesting improved chain mixing and reduced autocorrelation. However, at a larger sample size $n = 500$, the adaptive MCMC algorithm demonstrates lower inefficiency factors, indicating that it is more effective at navigating the increasingly complex posterior distribution. This outcome highlights the ability of the adaptive MCMC method to adjust to conditions where the posterior becomes more sharply peaked and exhibits stronger inter-parameter correlations. Such conditions render adaptive MCMC particularly effective, as also noted in [6].

Figure 1 provides spaghetti plots for the slope coefficient (β_1) across 100 replications under three error distributions (standard normal, right-skewed Gamma, heteroscedastic) and three sample sizes (100, 200, 500), since crossings for the intercept (β_0) were minimal. Each line connects estimated β_1 across quantiles, with red indicating a lower-quantile estimate exceeding a higher one indication non-monotonicity and blue indicating monotonicity. These plots offer a concise visual tally of monotonicity violations without inspecting individual replications. Frequentist quantile regression methods generate numerous red lines across all settings, demonstrating persistent quantile crossing. Gibbs sampling also fails to enforce ordering, particularly under skewed and heteroscedastic errors. In contrast, the proposed approach employing constrained priors yields exclusively blue lines across every scenario. This demonstrates that the constrained-prior framework consistently enforces monotonic quantile estimates regardless of error structure or sample size.

Table 1: Bias and RMSE of regression coefficients across quantiles and sample sizes under different error distributions.

n	Error[1]	Quantile	β_0			β_1		
			Frequentist[2]	Gibbs[2]	Adaptive MCMC[2]	Frequentist[2]	Gibbs[2]	Adaptive MCMC[2]
100	SN	0.10	0.099 (0.256)	0.005 (0.153)	0.054 (0.133)	0.027 (0.423)	0.290 (0.321)	0.022 (0.101)
		0.25	0.024 (0.160)	0.028 (0.098)	0.021 (0.091)	0.010 (0.272)	0.137 (0.175)	0.017 (0.108)
		0.50	0.004 (0.121)	0.000 (0.067)	0.009 (0.068)	0.049 (0.290)	0.019 (0.125)	0.001 (0.100)
		0.75	0.016 (0.150)	0.015 (0.091)	0.016 (0.086)	0.068 (0.375)	0.120 (0.174)	0.016 (0.114)
		0.90	0.110 (0.252)	0.015 (0.163)	0.044 (0.125)	0.086 (0.429)	0.295 (0.316)	0.027 (0.114)
	RSG	0.10	0.001 (0.004)	0.116 (0.122)	0.173 (0.177)	0.000 (0.005)	0.122 (0.134)	0.071 (0.079)
		0.25	0.012 (0.069)	0.006 (0.046)	0.008 (0.044)	0.007 (0.074)	0.056 (0.077)	0.045 (0.062)
		0.50	0.023 (0.054)	0.076 (0.088)	0.086 (0.096)	0.023 (0.087)	0.007 (0.055)	0.003 (0.056)
		0.75	0.102 (0.183)	0.077 (0.116)	0.081 (0.125)	0.049 (0.331)	0.109 (0.155)	0.025 (0.095)
		0.90	0.259 (0.660)	0.050 (0.196)	0.073 (0.187)	0.095 (0.774)	0.298 (0.338)	0.030 (0.137)
	H	0.10	0.138 (0.395)	0.058 (0.208)	0.049 (0.158)	0.079 (0.609)	0.296 (0.351)	0.022 (0.200)
		0.25	0.027 (0.182)	0.025 (0.091)	0.024 (0.082)	0.022 (0.437)	0.146 (0.224)	0.020 (0.154)
		0.50	0.000 (0.067)	0.001 (0.049)	0.002 (0.050)	0.038 (0.308)	0.033 (0.149)	0.035 (0.135)
		0.75	0.025 (0.153)	0.020 (0.096)	0.014 (0.096)	0.051 (0.352)	0.148 (0.207)	0.042 (0.171)
		0.90	0.147 (0.338)	0.014 (0.175)	0.055 (0.182)	0.039 (0.574)	0.283 (0.335)	0.063 (0.203)
200	SN	0.10	0.058 (0.119)	0.030 (0.095)	0.021 (0.091)	0.018 (0.234)	0.185 (0.218)	0.036 (0.099)
		0.25	0.020 (0.114)	0.000 (0.078)	0.002 (0.056)	0.001 (0.233)	0.087 (0.154)	0.018 (0.101)
		0.50	0.002 (0.073)	0.000 (0.050)	0.002 (0.051)	0.007 (0.199)	0.007 (0.110)	0.016 (0.114)
		0.75	0.002 (0.089)	0.004 (0.070)	0.009 (0.065)	0.026 (0.224)	0.094 (0.160)	0.047 (0.115)
		0.90	0.042 (0.137)	0.007 (0.109)	0.022 (0.092)	0.048 (0.237)	0.192 (0.238)	0.065 (0.125)
	RSG	0.10	0.000 (0.000)	0.037 (0.038)	0.114 (0.116)	0.000 (0.001)	0.023 (0.027)	0.048 (0.052)
		0.25	0.003 (0.009)	0.013 (0.017)	0.010 (0.021)	0.000 (0.012)	0.011 (0.024)	0.030 (0.039)
		0.50	0.013 (0.040)	0.051 (0.059)	0.067 (0.075)	0.006 (0.060)	0.004 (0.048)	0.007 (0.053)
		0.75	0.056 (0.146)	0.054 (0.108)	0.038 (0.088)	0.028 (0.229)	0.095 (0.152)	0.027 (0.103)
		0.90	0.196 (0.467)	0.009 (0.196)	0.035 (0.152)	0.034 (0.583)	0.263 (0.325)	0.019 (0.170)
	H	0.10	0.093 (0.277)	0.013 (0.146)	0.038 (0.121)	0.112 (0.437)	0.145 (0.266)	0.002 (0.163)
		0.25	0.011 (0.106)	0.003 (0.076)	0.007 (0.060)	0.016 (0.284)	0.091 (0.204)	0.009 (0.156)
		0.50	0.002 (0.038)	0.004 (0.029)	0.000 (0.033)	0.020 (0.254)	0.011 (0.155)	0.022 (0.154)
		0.75	0.026 (0.109)	0.016 (0.075)	0.008 (0.066)	0.049 (0.281)	0.059 (0.169)	0.017 (0.175)
		0.90	0.069 (0.244)	0.011 (0.149)	0.039 (0.124)	0.040 (0.426)	0.173 (0.284)	0.022 (0.175)
500	SN	0.10	0.019 (0.085)	0.012 (0.062)	0.012 (0.059)	0.026 (0.198)	0.059 (0.157)	0.035 (0.116)
		0.25	0.000 (0.052)	0.004 (0.042)	0.004 (0.045)	0.017 (0.145)	0.055 (0.128)	0.033 (0.109)
		0.50	0.002 (0.045)	0.002 (0.035)	0.003 (0.036)	0.014 (0.119)	0.015 (0.094)	0.009 (0.096)
		0.75	0.010 (0.052)	0.001 (0.042)	0.000 (0.042)	0.001 (0.149)	0.046 (0.118)	0.025 (0.098)
		0.90	0.020 (0.069)	0.010 (0.062)	0.023 (0.060)	0.022 (0.179)	0.097 (0.170)	0.052 (0.107)
	RSG	0.10	0.000 (0.000)	0.006 (0.007)	0.048 (0.048)	0.000 (0.000)	0.002 (0.004)	0.016 (0.019)
		0.25	0.001 (0.002)	0.009 (0.010)	0.012 (0.013)	0.000 (0.004)	0.001 (0.007)	0.009 (0.014)
		0.50	0.004 (0.012)	0.022 (0.025)	0.032 (0.035)	0.002 (0.033)	0.001 (0.028)	0.001 (0.026)
		0.75	0.021 (0.058)	0.028 (0.048)	0.034 (0.052)	0.004 (0.143)	0.035 (0.112)	0.022 (0.091)
		0.90	0.030 (0.159)	0.000 (0.098)	0.014 (0.091)	0.056 (0.347)	0.128 (0.247)	0.017 (0.147)
	H	0.10	0.036 (0.127)	0.012 (0.104)	0.005 (0.074)	0.069 (0.245)	0.129 (0.232)	0.025 (0.155)
		0.25	0.005 (0.053)	0.002 (0.041)	0.005 (0.041)	0.014 (0.189)	0.024 (0.157)	0.001 (0.135)
		0.50	0.001 (0.017)	0.000 (0.014)	0.003 (0.016)	0.013 (0.144)	0.009 (0.120)	0.017 (0.124)
		0.75	0.011 (0.048)	0.012 (0.035)	0.011 (0.042)	0.004 (0.173)	0.028 (0.139)	0.003 (0.132)
		0.90	0.023 (0.111)	0.004 (0.084)	0.009 (0.073)	0.017 (0.223)	0.071 (0.187)	0.039 (0.134)

[1] SN for Standard Normal; RSG for Right-Skewed Gamma; and H for Heteroscedastic
[2] |Bias| (RMSE)

Table 2: Inefficiency factor and Monte Carlo Standard Errors of regression coefficients across quantiles and sample sizes under different error distributions.

n	Error[1]	Quantile	β_0 Gibbs[2]	β_0 Adaptive MCMC[2]	β_1 Gibbs[2]	β_1 Adaptive MCMC[2]
100	SN	0.10	4.870 (0.004)	5.898 (0.005)	2.595 (0.003)	6.192 (0.003)
		0.25	3.431 (0.004)	5.273 (0.004)	1.926 (0.003)	7.363 (0.002)
		0.50	3.071 (0.003)	3.404 (0.003)	1.993 (0.002)	3.243 (0.003)
		0.75	3.406 (0.004)	5.040 (0.004)	2.132 (0.003)	6.782 (0.002)
		0.90	4.574 (0.004)	5.304 (0.005)	2.622 (0.003)	6.579 (0.003)
	RSG	0.10	4.596 (0.002)	7.310 (0.003)	3.479 (0.002)	6.388 (0.002)
		0.25	1.746 (0.001)	5.454 (0.002)	1.591 (0.002)	5.416 (0.002)
		0.50	2.240 (0.002)	4.130 (0.002)	1.614 (0.002)	3.998 (0.002)
		0.75	4.401 (0.003)	5.691 (0.004)	2.358 (0.002)	6.430 (0.002)
		0.90	5.717 (0.005)	5.939 (0.006)	2.872 (0.003)	7.442 (0.003)
	H	0.10	5.851 (0.004)	6.276 (0.005)	2.711 (0.003)	7.505 (0.002)
		0.25	3.855 (0.003)	6.109 (0.003)	2.314 (0.003)	6.041 (0.002)
		0.50	2.328 (0.002)	3.359 (0.003)	2.125 (0.003)	3.339 (0.003)
		0.75	3.399 (0.003)	5.905 (0.003)	2.262 (0.003)	6.955 (0.002)
		0.90	5.559 (0.004)	6.413 (0.005)	2.606 (0.003)	6.821 (0.002)
200	SN	0.10	5.905 (0.003)	4.222 (0.004)	4.121 (0.002)	5.396 (0.002)
		0.25	4.608 (0.002)	3.843 (0.003)	3.064 (0.002)	5.171 (0.002)
		0.50	4.188 (0.002)	3.348 (0.002)	2.889 (0.002)	2.925 (0.002)
		0.75	4.101 (0.002)	3.887 (0.003)	2.950 (0.002)	5.250 (0.002)
		0.90	5.573 (0.003)	4.732 (0.004)	4.030 (0.002)	6.104 (0.002)
	RSG	0.10	4.523 (0.001)	5.888 (0.002)	3.418 (0.001)	4.575 (0.002)
		0.25	1.796 (0.001)	3.557 (0.001)	1.729 (0.001)	3.715 (0.001)
		0.50	3.290 (0.001)	4.013 (0.002)	2.363 (0.001)	3.416 (0.001)
		0.75	6.528 (0.002)	4.455 (0.003)	3.814 (0.002)	5.042 (0.002)
		0.90	9.021 (0.004)	4.976 (0.005)	4.611 (0.003)	6.183 (0.002)
	H	0.10	8.029 (0.003)	4.364 (0.004)	4.330 (0.002)	5.166 (0.002)
		0.25	4.617 (0.002)	4.406 (0.002)	3.677 (0.002)	4.951 (0.002)
		0.50	3.091 (0.002)	2.963 (0.002)	2.981 (0.002)	2.989 (0.002)
		0.75	5.008 (0.002)	4.158 (0.002)	3.436 (0.002)	4.954 (0.002)
		0.90	7.467 (0.003)	4.397 (0.004)	4.339 (0.002)	5.523 (0.002)
500	SN	0.10	7.275 (0.002)	3.083 (0.002)	5.968 (0.001)	3.960 (0.002)
		0.25	5.379 (0.001)	3.208 (0.002)	4.565 (0.001)	3.416 (0.002)
		0.50	5.295 (0.001)	2.452 (0.002)	4.076 (0.001)	2.810 (0.001)
		0.75	6.192 (0.001)	3.182 (0.002)	4.865 (0.001)	3.667 (0.002)
		0.90	7.658 (0.001)	3.470 (0.002)	5.712 (0.001)	4.326 (0.002)
	RSG	0.10	3.067 (0.000)	6.508 (0.001)	2.600 (0.000)	5.006 (0.001)
		0.25	2.132 (0.000)	4.382 (0.001)	1.850 (0.000)	3.830 (0.001)
		0.50	4.848 (0.001)	4.043 (0.001)	3.415 (0.001)	3.869 (0.001)
		0.75	7.828 (0.001)	3.361 (0.002)	5.587 (0.001)	3.585 (0.001)
		0.90	10.313 (0.002)	3.841 (0.003)	7.723 (0.002)	4.395 (0.002)
	H	0.10	8.847 (0.002)	3.263 (0.003)	6.229 (0.001)	3.896 (0.002)
		0.25	6.602 (0.001)	3.176 (0.002)	4.912 (0.001)	3.973 (0.002)
		0.50	3.550 (0.001)	2.806 (0.001)	4.806 (0.001)	2.911 (0.002)
		0.75	6.334 (0.001)	2.946 (0.002)	5.093 (0.001)	3.618 (0.002)
		0.90	9.371 (0.002)	3.367 (0.003)	6.087 (0.001)	4.305 (0.002)

[1] SN for Standard Normal; RSG for Right-Skewed Gamma; and H for Heteroscedastic
[2] IF (MCSE)

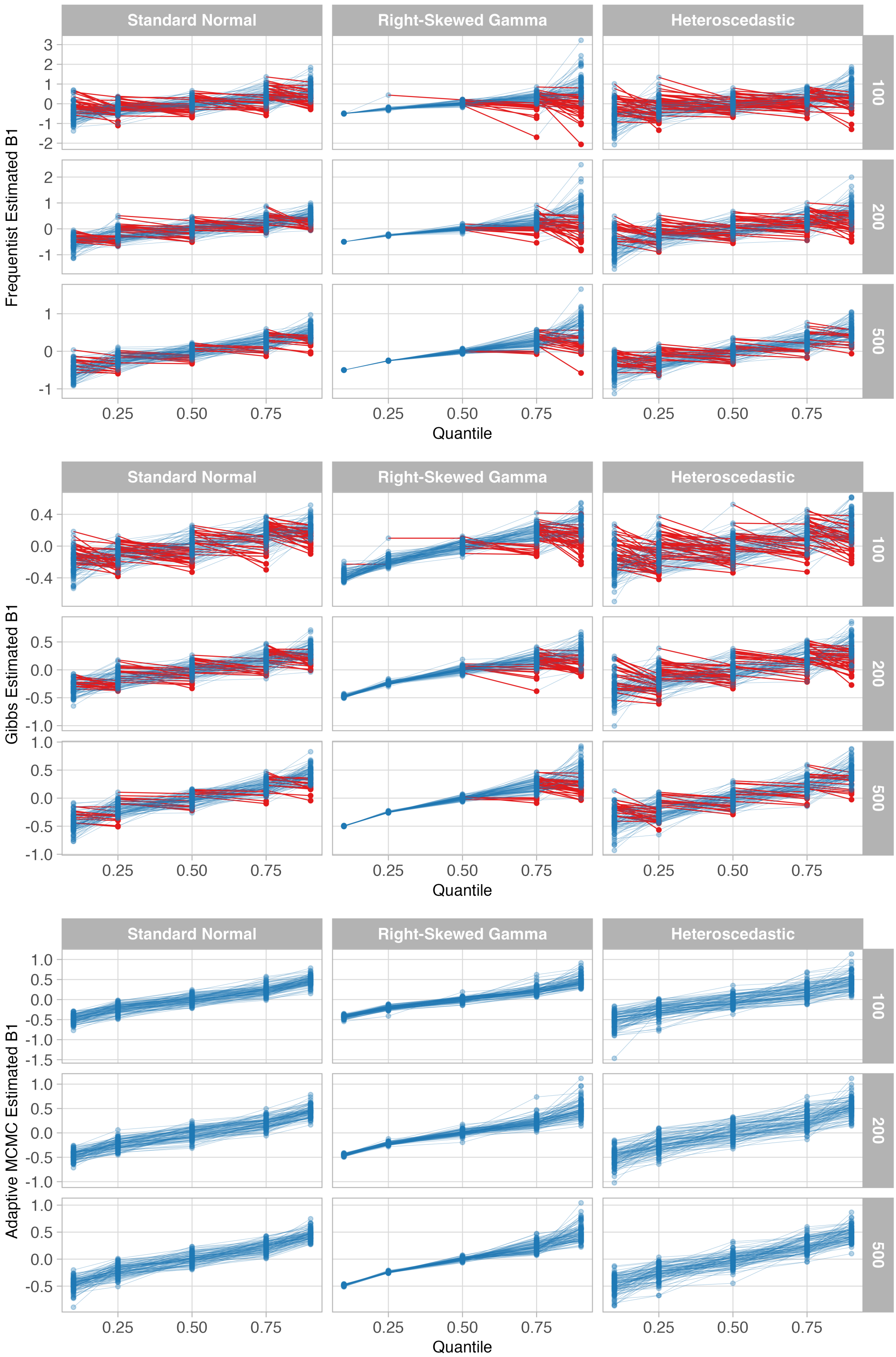

Figure 1: Slope coefficient β_1 monotonicity check for quantile crossing.

4 Application

In order to illustrate the proposed methodology using real-world data, the adaptive MCMC Bayesian quantile regression approach was applied to the Philippine Family Income and Expenditure Survey (FIES), a nationwide survey of households conducted triennially by the Philippine Statistics Authority (PSA). The 2023 dataset was employed to investigate the effect of household income on educational expenditures. For this analysis, the sample was restricted to households in Region 4 reporting non-zero educational spending, based on preliminary findings suggesting the presence of quantile crossing. A total of 4,083 observations met these inclusion criteria. A simple linear quantile regression model is fitted for quantiles $p = 0.1$ to 0.9, and is given by

$$\log(\text{Education}) = \beta_0 + \beta_1 \log(\text{Income})$$

To fit the model, 10,000 iterations were used, discarding the initial 5,000 iterations as a burn-in period. The same prior specifications adopted in the simulation study were also used here. The results below present a comparison between the frequentist estimation procedure and the proposed Bayesian approach.

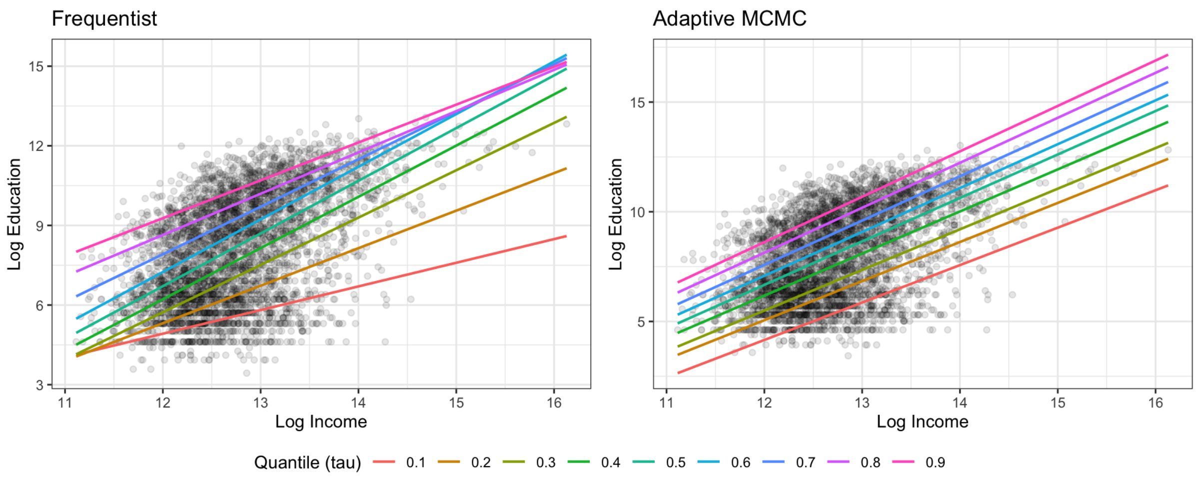

Figure 2: Fitted Quantile Regression Lines

Figure 2 shows the fitted quantile regression lines using both frequentist and adaptive MCMC Bayesian approaches. Both methods reveal a positive association between log income and log education expenditure, which is in line with results reported in the literature, see [1, 10]. However, the frequentist estimates exhibit quantile crossing, especially at the distribution tails. In contrast, the Bayesian estimates maintain proper ordering across quantiles, addressing the crossing issue. This highlights the improved interpretability and robustness of the adaptive MCMC approach in estimating conditional quantiles.

Table 3 presents a comparison of the objective function values defined in Equation 2 for both the frequentist and adaptive MCMC methods. Although the frequentist estimator is designed to minimize the objective function, the adaptive MCMC approach yields comparable values across most quantiles, with particularly close values around $p = 0.5$. Notably, larger discrepancies are observed at the extreme quantiles, where the Bayesian technique successfully addresses the issue of quantile-line crossing.

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Table 3: Objective function minimization results comparing frequentist and adaptive MCMC

Method	Quantile								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Frequentist	1261.022	2125.449	2678.813	2974.562	3042.276	2886.613	2517.305	1923.418	1117.779
aMCMC	1302.432	2138.102	2680.485	2975.563	3043.289	2887.062	2525.758	1956.306	1155.441

These findings indicate that the adaptive MCMC Bayesian quantile regression approach applied to the 2023 FIES data captures the positive relationship between household income and educational expenditures while effectively addressing the quantile crossing problem observed in the frequentist estimates. The presence of quantile crossing in the frequentist results, especially at the extremes, raises concerns about the validity of inferences for the lowest and highest income groups. In contrast, the adaptive MCMC method maintains the correct ordering of quantile curves and achieves reasonably accurate empirical coverage across the distribution, as shown in Table 3.

Although both methods approach nominal coverage rates, the frequentist estimates slightly outperform the Bayesian approach in achieving exact empirical coverage. However, the Bayesian method offers better curve ordering and stability—critical for drawing reliable policy conclusions. For instance, the consistently lower coverage at the 0.1 and 0.2 quantiles implies some underestimation of uncertainty for low-income households, which underscores the importance of incorporating robust prior information in future modeling.

From a policy perspective, the results suggest that households across all income levels increase educational spending as income rises, but the impact differs by income group. Well-ordered quantile estimates make it clear that lower-income households allocate a lesser proportion of income to education as income increases, suggesting that targeted subsidies or conditional cash transfers could be particularly effective for these groups. For higher-income households, the relationship remains positive but more stable, indicating that broad-based incentives may have limited marginal impact. These insights provide evidence-based guidance for policymakers aiming to design equitable educational support mechanisms.

5 Conclusion

This study introduced a Bayesian quantile regression framework that employs an adaptive MCMC approach and compared it against standard frequentist and Gibbs-based methods. By applying a constrained prior to each quantile parameter, monotonicity was rigorously enforced, eliminating the possibility of quantile crossing in linear settings. Empirical findings showed that the proposed method yields lower bias and RMSE while also exhibiting superior chain mixing, particularly at larger sample sizes. The approach was also applied to real-world data from the Philippine Family Income and Expenditure Survey (FIES), where the fitted quantile regression lines did not exhibit any quantile crossing and had results aligned with existing literature on household income and educational expenditure. These findings underscore the robustness and practical utility of this approach in multi-quantile regression, where preserving monotonicity across quantiles is essential for both interpretability and modeling accuracy, while also enabling fast mixing without compromising estimation accuracy.

This study directly supports Sustainable Development Goal 4, specifically Target 4.3, which aims for equal access to affordable and quality technical, vocational, and higher education. Using Bayesian quantile regression on education spending data from the Philippine Family Income and Expenditure Survey (FIES), the study clearly shows how household income affects education expenses at various spending levels. These results can guide policymakers in addressing inequalities in educational funding—particularly by targeting financial assistance to lower-income households, to make income gains translate more directly into increased educational investment which then helps ensure fairer and more inclusive access to education.

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