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θ_β-Open Sets and θ_β-Continuous Functions In The Product Space

Laika Angela B. Añora* & Mhelmar A. Labendia

Department of Mathematics and Statistics
MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines
laikaangela.anora@g.msuiit.edu.ph, mhelmar.labendia@g.msuiit.edu.ph

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Abstract

In this paper, we introduced and characterized a new class of open set called θ_β-open set. Notably, the collection of all θ_β-open sets forms a topology. We then examined the relationship between θ_β-open sets and other well-known concepts, such as classical open sets, θ-open sets, and β-open sets. Additionally, we defined and investigated the concepts of θ_β-interior and θ_β-closure of a set, as well as θ_β-open functions, θ_β-closed functions, θ_β-continuous functions, and θ_β-connectedness. Finally, we present characterizations of θ_β-continuous functions from an arbitrary topological space into the product space, along with some versions of separation axioms.

1 Introduction and Preliminaries

Over time, numerous mathematicians have been drawn to the idea of refining or expanding classical topological concepts by replacing them with alternatives that possess either weaker or stronger properties. This approach can be traced back to Levine [16] in 1963 when he introduced the concepts of semi-open and semi-closed sets, as well as semi-continuity for functions. This pioneering idea led to the development of new results, many of which serve as generalizations of established theories.

In 1968, Velicko [21] introduced the concepts of θ-closure and θ-interior for subsets of a topological space, and subsequently defined the notion of θ-continuity for functions in topological spaces. Several authors then have obtained results related to θ-open sets, see [1, 4, 5, 6, 7, 8].

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The θ-closure and θ-interior of A are, respectively, denoted and defined by

$$Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \varnothing \text{ for every open set } U \text{ containing } x\}$$

and

$$Int_\theta(A) = \{x \in X : Cl(U) \subseteq A \text{ for some open set } U \text{ containing } x\},$$

*Corresponding author

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where $Cl(U)$ is the closure of U in X. A subset A of X is θ-closed if $Cl_\theta(A) = A$ and θ-open if $Int_\theta(A) = A$. Equivalently, A is θ-open if and only if $X \setminus A$ is θ-closed. It is known that the collection $\mathcal{T}_\theta$ of all θ-open sets forms a topology on X, which is strictly coarser that $\mathcal{T}$.

In 1983, El-Monsef et al. [10] introduced the concept of β-open sets and expanded the theory by defining and characterizing β-continuous functions and β-open mappings. Several papers have studied the concepts of β-open sets and its corresponding topological concepts, such as [3, 11, 18].

A subset A of a topological space $(X, \mathcal{T})$ is said to be β-open if $A \subseteq Cl(Int(Cl(A)))$. The complement of β-open is called β-closed set. For a given subset A of a topological space $(X, \mathcal{T})$, $\beta Cl(A) = A \cup Int(Cl(Int(A)))$ [11]. Moreover, the union of all β-open sets in X that are contained in A is called β-interior of A and is denoted by $\beta Int(A)$ [11]. Equivalently, A is β-open (resp., β-closed) if and only if $A = \beta Int(A)$ (resp., $A = \beta Cl(A)$). It is worth noting that the collection of all β-open sets is not necessarily a topology on X.

A topological space $(X, \mathcal{T})$ is said to be connected (resp., θ-connected, β-connected [19]) if X cannot be written as the union of two nonempty disjoint open (resp., θ-open, β-open) sets. Otherwise, $(X, \mathcal{T})$ is disconnected (resp., θ-disconnected, β-disconnected). Furthermore, it has been shown in [13] that β-connected $\Rightarrow$ connected.

It is known that $Int_\theta(A)$ is the largest θ-open set contained in A and $Cl_\theta(A)$ is the smallest θ-closed set containing A [14]. Moreover, $x \in Int_\theta(A)$ if and only if there exists an open set U containing x such that $Cl(U) \subseteq A$ and $x \in Cl_\theta(A)$ (resp., $x \in \beta Cl(A)$) if and only if for every open set (resp., β-open set) U containing x, $Cl(U) \cap A \neq \varnothing$ [21] (resp., $U \cap A \neq \varnothing$ [11]).

Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\mathcal{T}_\alpha$ be the topology on Y_α. The Tychonoff topology on $\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $\langle U_\alpha \rangle = \mathrm{p}_\alpha^{-1}(U_\alpha)$, where $p_\alpha : \prod\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$, the αth coordinate projection map is defined by $p_\alpha(\langle Y_\beta \rangle) = y_\alpha$, U_α ranges over all members of $\mathcal{T}_\alpha$, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \cdots, \alpha_n$ and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}, U_{\alpha_2} \subseteq Y_{\alpha_2}, \cdots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_\alpha^{-1}(U_{\alpha_1}) \cap p_\alpha^{-1}(U_{\alpha_2}) \cap \cdots \cap p_\alpha^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, \cdots, B_{\alpha_k} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, \cdots, k\}$.

Now, the projection map $p_\alpha : \prod\{Y_\alpha : \alpha \in A\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in A$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in A\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α-th coordinate projection map.

In this paper, we introduced the concept of θ_β-open sets and explore their relationships with other well-established concepts in topology, including classical open sets, θ-open sets, and β-open sets. Some topological concepts related to θ_β-open sets are also defined and studied.

2 θ_β-Open and θ_β-Closed Functions

In this section, we define and characterize the concepts of θ_β-open and θ_β-closed functions. Throughout this paper, if no confusion arise, let X and Y be topological spaces.

Definition 2.1. Let X be a topological space. A subset A of X is said to be θ_β-*open* if for all $x \in A$, there exists an open set U containing x such that $\beta Cl(U) \subseteq A$. A subset F of X is said to be θ_β-*closed* if its complement $X \setminus F$ is θ_β-open.

Theorem 2.2. *Let X be a topological space and $A \subseteq X$. Then the following holds:*

(i) *If A is θ-open, then A is θ_β-open.*

(ii) *If A is θ_β-open, then A is open.*

Proof. (i) Let A be θ-open and let $x \in A$. Then there exists an open set U containing x such that $Cl(U) \subseteq A$. Moreover, $\beta Cl(U) \subseteq Cl(U) \subseteq A$, see [2, Diagram I]. Thus, A is θ_β-open.

(ii) Suppose that A is θ_β-open and let $x \in A$. By Definition 2.1, there exists an open set U containing x such that $\beta Cl(U) \subseteq A$. Observe that $U \subseteq \beta Cl(U) \subseteq A$. Hence, A is open. □

Corollary 2.3. *Let X be a topological space and $A \subseteq X$. Then the following holds:*

(i) *If A is θ-closed, then A is θ_β-closed.*

(ii) *If A is θ_β-closed, then A is closed.*

Proof. (i) Suppose that A is θ-closed. Then $X \setminus A$ is θ-open so that by Theorem 2.2 (i), $X \setminus A$ is θ_β-open. Thus, $X \setminus (X \setminus A) = A$ is θ_β-closed.

(ii) Assume that A is θ_β-closed. Then $X \setminus A$ is θ_β-open and by Theorem 2.2 (ii), $X \setminus A$ is open. Hence, $X \setminus (X \setminus A) = A$ is closed. □

In view of [2, Diagram I], Theorem 2.2, and Corollary 2.3, the following remark holds.

Remark 2.4. The following diagram holds for any subset of a topological space:

$$\theta\text{-open} \Longrightarrow \theta_\beta\text{-open} \Longrightarrow \text{open} \Longrightarrow \beta\text{-open}.$$

We note that the above diagram is also true for their respective closed sets. The reverse implications of Remark 2.4 are not true as shown in the next examples.

Example 2.5. Let X be a topological space given by $X = \{a, b, c, d\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. The sets $\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}$, and $\{b, c\}$ are θ_β-open but not θ-open.

Example 2.6. Consider again the topological space X in Example 2.5. Observe that $\{a, b, c\}$ is open but not θ_β-open.

Example 2.7. Consider the space X given in Example 2.5. Let $A = \{b, c, d\}$. Then A is β-open not open.

Example 2.8. Consider the real line $\mathbb{R}$ with the standard topology $\mathcal{T}_\mathbb{R}$. It is known that every open interval (a, b) where $a, b \in \mathbb{R}$ and $a < b$ is open in $\mathbb{R}$. Now, we will show that every open interval is θ-open.

Indeed, (a, b) is θ-open since for every $x \in (a, b)$, there exists $\varepsilon > 0$ such that

$$x \in (x - \varepsilon, x + \varepsilon) \subseteq Cl((x - \varepsilon, x + \varepsilon)) = [x - \varepsilon, x + \varepsilon] \subseteq (a, b).$$

Since every θ-open is θ_β-open by Theorem 2.2 (i), (a, b) is also θ_β-open. Therefore, $\mathcal{T}_\mathbb{R} = \mathcal{T}_\theta = \mathcal{T}_{\theta_\beta}$ in $\mathbb{R}$ with the usual topology.

The following result is likely known, but despite a thorough search of the literature, we were unable to find a suitable reference. For the sake of completeness, we provide the statement and proof below.

Lemma 2.9. *Let X be a topological space and $A, B \subseteq X$. Then the following statements hold:*

(i) *If $A \subseteq B$, then $\beta Cl(A) \subseteq \beta Cl(B)$.*

(ii) *$\beta Cl(A \cap B) \subseteq \beta Cl(A) \cap \beta Cl(B)$.*

Proof. (i) Let $A \subseteq B$ and $x \in \beta Cl(A)$. Then for every β-open set U containing x, $U \cap A \neq \varnothing$. Since $A \subseteq B$, $U \cap B \neq \varnothing$. Thus, $x \in \beta Cl(B)$. Accordingly, $\beta Cl(A) \subseteq \beta Cl(B)$.

(ii) Because $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we have $\beta Cl(A \cap B) \subseteq \beta Cl(A)$ and $\beta Cl(A \cap B) \subseteq \beta Cl(B)$ by (i). It follows that $\beta Cl(A \cap B) \subseteq \beta Cl(A) \cap \beta Cl(B)$. □

Remark 2.10. Let X be a topological space. Then

(i) The arbitrary union of θ_β-open sets is θ_β-open.

(ii) The finite intersection of θ_β-open sets is θ_β-open.

Proof. (i) Let $\{O_\alpha : \alpha \in \mathcal{A}\}$ be a collection of θ_β-open subsets of X and let $x \in \bigcup_{\alpha \in \mathcal{A}} O_\alpha$. Then $x \in O_{\alpha_0}$ for some $\alpha_0 \in \mathcal{A}$. Since O_{α_0} is θ_β-open, there exists an open set F_{α_0} containing x such that $\beta Cl(F_{\alpha_0}) \subseteq O_{\alpha_0} \subseteq \bigcup_{\alpha \in \mathcal{A}} O_\alpha$. Thus, $\bigcup_{\alpha \in \mathcal{A}} O_\alpha$ is θ_β-open.

(ii) Let G_1 and G_2 be θ_β-open sets and $x \in G_1 \cap G_2$. Then there exist open sets U_1 and U_2 with $x \in U_1 \cap U_2$ such that $\beta Cl(U_1) \subseteq G_1$ and $\beta Cl(U_2) \subseteq G_2$. Now, $U_1 \cap U_2$ is an open set that contains x so that

$$\beta Cl(U_1 \cap U_2) \subseteq \beta Cl(U_1) \cap \beta Cl(U_2) \subseteq G_1 \cap G_2$$

by Lemma 2.9. Hence, $G_1 \cap G_2$ is θ_β-open and the result follows. □

Hence, in view of Remark 2.10, the family of all θ_β-open subsets of a topological space X forms a topology on X, denoted by $\mathcal{T}_{\theta_\beta}$.

Corollary 2.11. *Let X be a topological space. Then*

(i) *The arbitrary intersection of θ_β-closed sets is θ_β-closed.*

(ii) *The finite union of θ_β-closed sets is θ_β-closed.*

Proof. (i) Let $\{F_\alpha : \alpha \in \mathcal{A}\}$ be a family of θ_β-closed sets in X. Then $X \setminus F_\alpha$ is θ_β-open for all $\alpha \in \mathcal{A}$. By Remark 2.10 (i), $X \setminus \bigcap_{\alpha \in \mathcal{A}} F_\alpha = \bigcup_{\alpha \in \mathcal{A}} (X \setminus F_\alpha)$ is θ_β-open. Therefore, $\bigcap_{\alpha \in \mathcal{A}} F_\alpha$ is θ_β-closed.

(ii) Let F_1 and F_2 be θ_β-closed sets in X. Then $X \setminus F_1$ and $X \setminus F_2$ are θ_β-open. Since $X \setminus (F_1 \cup F_2) = (X \setminus F_1) \cap (X \setminus F_2)$ is θ_β-open by Remark 2.10 (ii), $F_1 \cup F_2$ is θ_β-closed. Therefore, the conclusion hold. □

Theorem 2.12. *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Assume that A is open. If A is β-closed, then A is θ_β-open. In this case, $\mathcal{T} = \mathcal{T}_{\theta_\beta}$.*

Proof. Let A be both open and β-closed. Then for all $x \in A$, we have $x \in A \subseteq \beta Cl(A) = A \subseteq A$. Hence, A is θ_β-open and the claim holds. □

Definition 2.13. Let X be a topological space and $A \subseteq X$.

(i) The θ_β-interior of A, denoted by $Int_{\theta_\beta}(A)$, is defined by $Int_{\theta_\beta}(A) = \bigcup\{U : U$ is a θ_β-open set and $U \subseteq A\}$. By Remark 2.10, $Int_{\theta_\beta}(A)$ is the largest θ_β-open set contained in A. Moreover, $x \in Int_{\theta_\beta}(A)$ if and only if there exists a θ_β-open sets U containing x such that $U \subseteq A$.

(ii) The θ_β-closure of A denoted by $Cl_{\theta_\beta}(A)$, is defined by $Cl_{\theta_\beta}(A) = \bigcap\{F : F$ is a θ_β-closed set and $A \subseteq F\}$. In view of Remark 2.10, $Cl_{\theta_\beta}(A)$ is the smallest θ_β-closed set containing A. Moreover, $x \in Cl_{\theta_\beta}(A)$ if and only if for every θ_β-open set U containing x, $U \cap A \neq \varnothing$.

The statements in the succeeding remark can be obtained using arguments similar to those in classical topology and are therefore omitted.

Remark 2.14. Let X be a topological space and $A, B \subseteq X$. Then the following statements hold:

(i) $Int_{\theta_\beta}(A) \subseteq A$.

(ii) $A \subseteq B$ implies that $Int_{\theta_\beta}(A) \subseteq Int_{\theta_\beta}(B)$.

(iii) A is θ_β-open if and only if $A = Int_{\theta_\beta}(A)$.

(iv) $Int_{\theta_\beta}(A) = Int_{\theta_\beta}(Int_{\theta_\beta}(A))$.

(v) $Int_{\theta_\beta}(A \cap B) = Int_{\theta_\beta}(A) \cap Int_{\theta_\beta}(B)$.

(vi) $A \subseteq Cl_{\theta_\beta}(A)$.

(vii) $A \subseteq B$ implies that $Cl_{\theta_\beta}(A) \subseteq Cl_{\theta_\beta}(B)$.

(viii) A is θ_β-closed if and only if $A = Cl_{\theta_\beta}(A)$.

(ix) $Cl_{\theta_\beta}(A) = Cl_{\theta_\beta}(Cl_{\theta_\beta}(A))$.

(x) $Cl_{\theta_\beta}(A \cup B) = Cl_{\theta_\beta}(A) \cup Cl_{\theta_\beta}(B)$.

(xi) $Int_{\theta_\beta}(X \setminus A) = X \setminus Cl_{\theta_\beta}(A)$.

(xii) $Cl_{\theta_\beta}(X \setminus A) = X \setminus Int_{\theta_\beta}(A)$.

(xiii) $x \in Int_{\theta_\beta}(A)$ if and only if there exists an open set U containing x such that $\beta Cl(U) \subseteq A$.

(xiv) $x \in Cl_{\theta_\beta}(A)$ if and only if for every open set U containing x, $\beta Cl(U) \cap A \neq \varnothing$.

(xv) $Int_\theta(A) \subseteq Int_{\theta_\beta}(A) \subseteq Int(A) \subseteq A$.

(xvi) $A \subseteq Cl(A) \subseteq Cl_{\theta_\beta}(A) \subseteq Cl_\theta(A)$.

We shall give some characterizations of θ_β-open and θ_β-closed functions.

Definition 2.15. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *θ_β-open* on X if $f(G)$ is θ_β-open in Y for every open set G in X.

Example 2.16. Consider $X = \{a, b, c\}$ with the topology $\mathcal{T}_X = \{\varnothing, X, \{a, c\}, \{b\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\mathcal{T}_Y = \{\varnothing, Y, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}$. Define $f : X \to Y$ by $f = \{(a, 1), (b, 2), (c, 3)\}$. Note that the open sets in X are $\varnothing, X, \{a, c\}$, and $\{b\}$. Also, $f(\varnothing) = \varnothing, f(X) = Y, f(\{a, c\}) = \{1, 3\}$, and $f(\{b\}) = \{2\}$. Clearly, $\varnothing$ and Y are θ_β-open in Y. Moreover, since $\{2\}$ is open in Y and $\beta Cl(\{2\}) = \{2\} \subseteq \{2\}$, $\{2\}$ is θ_β-open in Y. Similarly, $\beta Cl(\{1, 3\}) = \{1, 3\} \subseteq \{1, 3\}$ so that $\{1, 3\}$ is θ_β-open in Y. This implies that $\varnothing, Y, \{2\}$, and $\{1, 3\}$ are all θ_β-open in Y. Thus, f is θ_β-open function on X.

Definition 2.17. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *θ_β-closed* on X if $f(F)$ is θ_β-closed in Y for every closed set F in X.

Example 2.18. Consider $X = \{a, b, c\}$ with the topology $\mathcal{T}_X = \{\varnothing, X, \{a, b\}, \{c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\mathcal{T}_Y = \{\varnothing, Y, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}$. Define $f : X \to Y$ by $f = \{(a, 3), (b, 1), (c, 2)\}$. Note that the closed sets in X are $\varnothing, X, \{c\}$, and $\{a, b\}$. Also, $f(\varnothing) = \varnothing, f(X) = Y, f(\{c\}) = \{2\}$, and $f(\{a, b\}) = \{1, 3\}$. Clearly, $\varnothing$ and Y are θ_β-closed in Y. Furthermore, since $\{2\}$ is open in Y and $\beta Cl(\{2\}) = \{2\} \subseteq \{2\}$, $\{2\}$ is θ_β-open in Y. Thus, $Y \setminus \{2\} = \{1, 3\}$ is θ_β-closed in Y. Similarly, $\beta Cl(\{1, 3\}) = \{1, 3\} \subseteq \{1, 3\}$ so that $\{1, 3\}$ is θ_β-open in Y which implies that $Y \setminus \{1, 3\} = \{2\}$ is θ_β-closed in Y. Therefore, $\varnothing, Y, \{1, 3\}$, and $\{2\}$ are all θ_β-closed in Y so that f is θ_β-closed function on X.

In view of Remark 2.4, and Definitions 2.15 and 2.17, we have the following remark.

Remark 2.19. The following diagram holds for a function $f : X \to Y$:

$$\begin{array}{ccc} \theta\text{-open function} & \Longrightarrow & \theta_\beta\text{-open function} \\ & & \Downarrow \\ \beta\text{-open function} & \Longleftarrow & \text{open function} \end{array}$$

Note that the diagram is also true for their respective closed functions. Moreover, the reverse implications of Remark 2.19 are not necessarily true as shown in the subsequent examples.

Example 2.20. Let $X = \{1, 2, 3, 4\}$ with topology $\mathcal{T}_X = \{\varnothing, X, \{1, 2\}, \{3, 4\}\}$ and $Y = \{a, b, c, d\}$ with topology $\mathcal{T}_Y = \{\varnothing, Y, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define a function $f : X \to Y$ by $f = \{(1, a), (2, c), (3, b), (4, d)\}$. Then f is θ_β-open on X but not θ-open since $f(\{1, 2\}) = \{a, c\}$ and $f(\{3, 4\}) = \{b, d\}$ are not θ-open in Y.

Example 2.21. Consider $X = \{0, 1, 2\}$ with topology $\mathcal{T}_X = \{\varnothing, X, \{0\}, \{1\}, \{0, 1\}\}$ and $Y = \{i, o, u\}$ with topology $\mathcal{T}_Y = \{\varnothing, Y, \{i\}, \{o\}, \{i, o\}\}$. Define $f : X \to Y$ by $f = \{(0, i), (1, o), (2, u)\}$. Then f is open on X but not θ_β-open since $f(\{0, 1\}) = \{i, o\}$ is not θ_β-open on Y.

Example 2.22. Consider $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$ with respective topologies $\mathcal{T}_X = \{\varnothing, X, \{1, 2\}, \{3, 4\}\}$ and $\mathcal{T}_Y = \{\varnothing, Y, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define a function $f = \{(1, c), (2, a), (3, b), (4, d)\}$. Then f is β-open on X but not open on X since $f(\{3, 4\}) = \{b, d\}$ is not open on Y.

Theorem 2.23. *Let X and Y be topological spaces and $f : X \to Y$ be a bijective function. Then f is θ_β-open if and only if f is θ_β-closed.*

Proof. Suppose that f is θ_β-open on X and let F be closed on X. Then $X \setminus F$ is open in X and $f(X \setminus F)$ is θ_β-open in Y. Since f is bijective, $f(X \setminus F) = Y \setminus f(F)$ is θ_β-open in Y, that is $f(F)$ is θ_β-closed in Y.

Conversely, assume that f is θ_β-closed on X and let G be open on X. Then $X \setminus G$ is closed on X and $f(X \setminus G)$ is θ_β-closed in Y. Since f is bijective, $f(X \setminus G) = Y \setminus f(G)$ is θ_β-closed in Y, that is, $f(G)$ is θ_β-open in Y. □

Theorem 2.24. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent*

(i) *f is θ_β-open on X;*

(ii) *$f(Int(A)) \subseteq Int_{\theta_\beta}(f(A))$ for each $A \subseteq X$; and*

(iii) *$f(B)$ is θ_β-open for every basic open set B in X.*

(iv) *For each $x \in X$ and for every open set O in X containing x, there exists an open set W in Y containing $f(x)$ such that $\beta Cl(W) \subseteq f(O)$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that f is θ_β-open on X. Then $f(G)$ is θ_β-open for all open set $G \subseteq X$. Let $A \subseteq X$. Observe that $Int(A) \subseteq A$ so that $f(Int(A)) \subseteq f(A)$. Since $Int(A)$ is open, $f(Int(A))$ is θ_β-open and is contained in $f(A)$. Note that $Int_{\theta_\beta}(f(A))$ is the largest θ_β-open set contained in $f(A)$ by Definition 2.13 (i). Thus, $f(Int(A)) \subseteq Int_{\theta_\beta}(f(A))$.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds. Let B be a basic open set in X. Then B is an open set in X and $B = Int(B)$. By assumption,

$$f(B) = f(Int(B)) \subseteq Int_{\theta_\beta}(f(B)) \subseteq f(B).$$

Hence, $f(B) = Int_{\theta_\beta}(f(B))$. Therefore, $f(B)$ is θ_β-open by Remark 2.14 (iii).

(iii) $\Rightarrow$ (iv) Suppose that (iii) holds. Let $x \in X$ and let O be an open set in X containing x. Then there exists a basic open set B containing x such that $x \in B \subseteq O$. This implies that $f(x) \in f(B) \subseteq f(O)$. Since $f(B)$ is θ_β-open, there exists an open set W in Y containing $f(x)$ such that $\beta Cl(W) \subseteq f(B) \subseteq f(O)$.

(iv) $\Rightarrow$ (i) Assume that (iv) holds. Let O be open in X and $y \in f(O)$. Then there exists $x \in O$ such that $f(x) = y$. By assumption, there exists an open set W in Y containing $f(x) = y$ such that $\beta Cl(W) \subseteq f(O)$. Hence, $f(O)$ is θ_β-open. □

Theorem 2.25. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ_β-closed in X.*

(ii) *$Cl_{\theta_\beta}(f(A)) \subseteq f(Cl(A))$ for every $A \subseteq X$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that f is θ_β-closed in X. Let $A \subseteq X$. Observe that $A \subseteq Cl(A)$ so that $f(A) \subseteq f(Cl(A))$. Since $Cl(A)$ is closed, $f(Cl(A))$ is θ_β-closed containing $f(A)$. Moreover, since $Cl_{\theta_\beta}(f(A))$ is the smallest θ_β-closed set containing $f(A)$, we have $Cl_{\theta_\beta}(f(A)) \subseteq f(Cl(A))$.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds. Let F be closed in X. Then $F = Cl(F)$. By assumption,

$$f(F) \subseteq Cl_{\theta_\beta}(f(F)) \subseteq f(Cl(F)) = f(F),$$

that is, $f(F) = Cl_{\theta_\beta}(f(F))$. Thus, $f(F)$ is θ_β-closed. □

Theorem 2.26. *Let X, Y, and Z be topological spaces. If $f : X \to Y$ is open on X and $g : Y \to Z$ is θ_β-open on Y, then the composition $g \circ f : X \to Z$ is θ_β-open on X.*

Proof. Let $x \in X$ and let U be open in X with $x \in U$. Since f is open on X, $f(U)$ is open on Y. This means that there exists an open set V_Y in Y containing $f(x)$ such that $V_Y \subseteq f(U)$. Since g is θ_β-open on Y, there exists an open set V_Z in Z containing $g(f(x)) = (g \circ f)(x)$ such that $\beta Cl(V_Z) \subseteq g(V_Y)$, by Theorem 2.24 (iv). Hence,

$$\beta Cl(V_Z) \subseteq g(V_Y) \subseteq g(f(U)) = (g \circ f)(U).$$

Therefore, by Theorem 2.24 (iv), $g \circ f$ is θ_β-open on X. □

Theorem 2.27. *Let X and Y be topological spaces and $\mathfrak{T}_A$ be the subspace topology on $A \subseteq X$. If $f : X \to Y$ is θ_β-open on X and A is open on X, then $f|_A : A \to Y$ is θ_β-open on A.*

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Proof. Let $x \in A$ and G be open in A containing x. Then $G = A \cap U$, where U is open in X. Since A is open in X, G is also open in X. By assumption, $f(G)$ is θ_β-open in Y containing $f(x)$, that is, there exists an open set V in Y containing $f(x) = f|_A(x)$ such that

$$\beta Cl(V) \subseteq f(G) = f|_A(G).$$

By Theorem 2.24 (iv), $f|_A : A \to Y$ is θ_β-open on A. □

Theorem 2.28. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $\mathcal{T}_B$ and $\mathcal{T}_C$ be two respective subspace topologies on $B, C \subseteq X$. If $X = B \cup C$ and $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a function such that $f|_B : (B, \mathcal{T}_B) \to (Y, \mathcal{T}_Y)$ and $f|_C : (C, \mathcal{T}_C) \to (Y, \mathcal{T}_Y)$ are θ_β-open, then $f : X \to Y$ is θ_β-open on X.*

Proof. Let $x \in X$ and $U \in \mathcal{T}_X$ containing x. Since $X = B \cup C$, it follows that $x \in B$ or $x \in C$. If $x \in B$, then $x \in B \cap U \in \mathcal{T}_B$. By assumption, there exists an open set W in Y containing $f|_B(x) = f(x)$ such that

$$\beta Cl(W) \subseteq f|_B(B \cap U) \subseteq f(U).$$

Hence, $f : X \to Y$ is θ_β-open by Theorem 2.24 (iv).

If $x \in C$, then $C \cap U \in \mathcal{T}_C$ with $x \in C \cap U$. By a similar argument, $f : X \to Y$ is θ_β-open on X. □

3 θ_β-Continuous Functions

This section characterizes the concept of θ_β-continuous functions and determines its relationship to the other versions of continuity.

Definition 3.1. Let X and Y be topological spaces. A function $f : X \to Y$ is *θ_β-continuous* on X if $f^{-1}(U)$ is θ_β-open for every open U in Y.

By Remark 2.4, we have the following remark.

Remark 3.2. The following diagram holds for a function $f : X \to Y$:

$$\begin{array}{ccc} \theta\text{-continuous} & \Longrightarrow & \theta_\beta\text{-continuous} \\ & & \Downarrow \\ \beta\text{-continuous} & \Longleftarrow & \text{continuous} \end{array}$$

The following examples illustrate that the reverse implications of Remark 3.2 do not hold.

Example 3.3. Let $X = \{a, b, c, d\}$ with topology $\mathcal{T}_X = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $Y = \{r, s, t, u\}$ with topology $\mathcal{T}_Y = \{\varnothing, Y, \{r, s\}, \{t, u\}\}$. Define $f : X \to Y$ by $f = \{(a, r), (b, t), (c, s), (d, u)\}$. Then $f^{-1} = \{(r, a), (t, b), (s, c), (u, d)\}$. Now, f is θ_β-continuous on X but not θ-continuous on X since $f^{-1}(\{r, s\}) = \{a, c\}$ and $f^{-1}(\{t, u\}) = \{b, d\}$ are not θ-open in X.

Example 3.4. Consider $X = \{i, o, u\}$ with topology $\mathcal{T}_X = \{\varnothing, X, \{i\}, \{o\}, \{i, o\}\}$ and $Y = \{0, 1, 2\}$ with topology $\mathcal{T}_Y = \{\varnothing, Y, \{0\}, \{1\}, \{0, 1\}\}$. Define a function $f : X \to Y$ by $f = \{(i, 0), (o, 1), (u, 2)\}$. Then $f^{-1} = \{(0, i), (1, o), (2, u)\}$. Note that f is continuous on X but not θ_β-continuous on X since $f^{-1}(\{0, 1\}) = \{i, o\}$ is not θ_β-open in X.

Example 3.5. Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$ with respective topologies given by $\mathcal{T}_X = \{\varnothing, Y, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{T}_Y = \{\varnothing, X, \{1, 2\}, \{3, 4\}\}$. Define a function $f = \{(c, 1), (a, 2), (b, 3), (d, 4)\}$. Then $f^{-1} = \{(1, c), (2, a), (3, b), (4, d)\}$. Observe that f is β-continuous on X but not continuous on X since $f^{-1}(\{3, 4\}) = \{b, d\}$ is not open on X.

Theorem 3.6. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ_β-continuous on X.*

(ii) *$f^{-1}(F)$ is θ_β-closed in X for each closed subset F of Y.*

(iii) *$f^{-1}(B)$ is θ_β-open for each (subbasic) basic open set B in Y.*

(iv) *For every $x \in X$ and every open set V of Y containing $f(x)$, there exists a θ_β-open set U containing x such that $f(U) \subseteq V$.*

(v) *$f(Cl_{\theta_\beta}(A)) \subseteq Cl(f(A))$ for each $A \subseteq X$.*

(vi) *$Cl_{\theta_\beta}(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for each $B \subseteq Y$.*

Proof. (i) $\Rightarrow$ (ii) Assume that f is θ_β-continuous on X. Let F be closed in Y. Then $Y \setminus F$ is open. Since f is θ_β-continuous, $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is θ_β-open in X. Thus, $f^{-1}(F)$ is θ_β-closed in X.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds and let O be open in Y. Then $Y \setminus O$ is closed. By assumption, $f^{-1}(Y \setminus O) = X \setminus f^{-1}(O)$ is θ_β-closed in X. It follows that $f^{-1}(O)$ is θ_β-open in X so that f is θ_β-continuous on X.

(i) $\Rightarrow$ (iii) Assume that f is θ_β-continuous on X. Since (subbasic) basic open sets are open, (iii) holds.

(iii) $\Rightarrow$ (i) Assume that (iii) holds. Let G be an open set in Y. Then $G = \bigcup\{B : B \in \mathcal{B}^*\}$ where $\mathcal{B}^* \subseteq \mathcal{B}$ is a basis for a topology in Y. This implies that

$$f^{-1}(G) = \bigcup\{f^{-1}(B) : B \in \mathcal{B}^*\},$$

where $f^{-1}(B)$ are θ_β-open sets on X by assumption. By Remark 2.10 (i), the arbitrary union of all θ_β-open sets is θ_β-open. Then $f^{-1}(G)$ is θ_β-open in X. Consequently, f is θ_β-continuous on X.

(i) $\Rightarrow$ (iv) Suppose that f is θ_β-continuous on X. Let $x \in X$ and let V be an open set in Y containing $f(x)$. Since f is θ_β-continuous, $f^{-1}(V)$ is θ_β-open in X containing x. Set $U = f^{-1}(V)$. Then $f(U) = f(f^{-1}(V)) \subseteq V$.

(iv) $\Rightarrow$ (v) Assume that (iv) holds. Let $A \subseteq X$ and $x \in Cl_{\theta_\beta}(A)$. Let G be an open set in Y containing $f(x)$. By assumption, there exists a θ_β-open set U in X containing x such that $f(U) \subseteq G$. Since $x \in Cl_{\theta_\beta}(A)$, $U \cap A \neq \varnothing$ by Definition 2.13 (ii). Thus,

$$\varnothing \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq G \cap f(A).$$

It follows that $f(x) \in Cl(f(A))$. Accordingly, $f(Cl_{\theta_\beta}(A)) \subseteq Cl(f(A))$.

(v) $\Rightarrow$ (vi) Let $B \subseteq Y$ and let $A = f^{-1}(B)$. Then $f(A) = f(f^{-1}(B)) \subseteq B$. By assumption, $f(Cl_{\theta_\beta}(A)) \subseteq Cl(f(A))$. Hence,

$$Cl_{\theta_\beta}(f^{-1}(B)) \subseteq f^{-1}(f(Cl_{\theta_\beta}(A))) \subseteq f^{-1}(Cl(f(A))) \subseteq f^{-1}(Cl(B)).$$

(vi) $\Rightarrow$ (ii) Let F be a closed subset of Y. Then $F = Cl(F)$. By assumption,

$$Cl_{\theta_\beta}(f^{-1}(F)) \subseteq f^{-1}(Cl(F)) = f^{-1}(F) \subseteq Cl_{\theta_\beta}(f^{-1}(F)).$$

Thus, $f^{-1}(F) = Cl_{\theta_\beta}(f^{-1}(F))$. By Remark 2.14 (viii), $f^{-1}(F)$ is θ_β-closed in X. □

Theorem 3.7. *Let X and Y be topological spaces and $f_A : X \to \mathcal{D}$ the characteristic function of a subset A of X, where $\mathcal{D}$ is the set $\{0,1\}$ with discrete topology. Then f_A is θ_β-continuous if and only if A is both θ_β-open and θ_β-closed.*

Proof. Suppose that A is both θ_β-open and θ_β-closed. Let U be an open set in $\{0,1\}$. Then

$$f_A^{-1}(U) = \begin{cases} \varnothing & \text{if } U = \varnothing, \\ X & \text{if } U = \{0,1\}, \\ A & \text{if } U = \{1\}, \\ X \setminus A & \text{if } U = \{0\}. \end{cases}$$

Hence, f_A^{-1} is θ_β-open and so f_A is θ_β-continuous.

Conversely, assume that f_A is θ_β-continuous. Let $U_1 = \{1\}$ and $U_2 = \{0\}$. Then U_1 and U_2 are both open in $\{0,1\}$. Thus, $f_A^{-1}(U_1) = A$ and $f_A^{-1}(U_2) = X \setminus A$ are θ_β-open in X. Accordingly, A is both θ_β-open and θ_β-closed. □

Theorem 3.8. *Let X, Y, and Z be topological spaces. If $f : X \to Y$ is θ_β-continuous on X and $g : Y \to Z$ is continuous on Y, then the composition $g \circ f : X \to Z$ is θ_β-continuous on X.*

Proof. Let U be open in Z. Since g is continuous on Y, $g^{-1}(U)$ is open on Y. By assumption, f is θ_β-continuous on X so that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is θ_β-open on X. Thus, $g \circ f$ is θ_β-continuous on X. □

4 θ_β-Continuous Functions in the Product Space

The following results are related to θ_β-continuous functions from an arbitrary topological space into the product space.

In the succeeding results, if $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ is a product space and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, we denote $A_{\alpha_1} \times \cdots \times A_{\alpha_n} \times \prod\{Y_\alpha : \alpha \notin K\}$ by $\langle A_{\alpha_1}, \ldots, A_{\alpha_n} \rangle$, $K = \{\alpha_1, \ldots, \alpha_n\}$. If $Y = \prod\{Y_i : 1 \leq i \leq n\}$ is a finite product, we denote $A_1 \times \cdots \times A_n$ by $\langle A_1, \ldots, A_n \rangle$.

Theorem 4.1. *Let $Y = \prod\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $\varnothing \neq O_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then $O = \langle O_1, \ldots, O_n \rangle$ is β-open if and only if each O_i is β-open.*

Proof. Suppose that $O = \langle O_1, \ldots, O_n \rangle$ is β-open. Then

$$\begin{aligned} O &\subseteq Cl(Int(Cl(O))) \\ &= Cl(Int(Cl(\langle O_1, \ldots, O_n \rangle))) \\ &= Cl(Int(\langle Cl(O_1), \ldots, Cl(O_n) \rangle)) \\ &= Cl(\langle Int(Cl(O_1)), \ldots, Int(Cl(O_n)) \rangle) \\ &= \langle Cl(Int(Cl(O_1))), \ldots, Cl(Int(Cl(O_n))) \rangle. \end{aligned}$$

Hence, for every $i = 1, \ldots, n$, $O_i \subseteq Cl(Int(Cl(O_i)))$. Therefore, each O_i is β-open.

Conversely, assume that each O_i is β-open. Then for every $i = 1, \ldots, n$, $O_i \subseteq Cl(Int(Cl(O_i)))$. Thus,

$$\begin{aligned} O &= \langle O_1, \ldots, O_n \rangle \\ &\subseteq \langle Cl(Int(Cl(O_1))), \ldots, Cl(Int(Cl(O_n))) \rangle \\ &= Cl(\langle Int(Cl(O_1)), \ldots, Int(Cl(O_n)) \rangle) \\ &= Cl(Int(\langle Cl(O_1), \ldots, Cl(O_n) \rangle)) \\ &= Cl(Int(Cl(\langle O_1, \ldots, O_n \rangle))) \\ &= Cl(Int(Cl(O))). \end{aligned}$$

Therefore, O is β-open. □

Theorem 4.2. *Let $Y = \prod\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then*

$$\beta Cl(\langle A_1, \ldots, A_n\rangle) \subseteq \langle \beta Cl(A_1), \ldots, \beta Cl(A_n)\rangle.$$

Proof. Observe that

$$\begin{aligned}
\beta Cl(\langle A_1, \ldots, A_n\rangle) &= \langle A_1, \cdots, A_n\rangle \cup Int(Cl(Int(\langle A_1, \cdots, A_n\rangle)))\\
&= \langle A_1, \cdots, A_n\rangle \cup Int(Cl(\langle Int(A_1), \ldots, Int(A_n)\rangle))\\
&= \langle A_1, \cdots, A_n\rangle \cup Int(\langle Cl(Int(A_1)), \ldots Cl(Int(A_n))\rangle)\\
&= \langle A_1, \cdots, A_n\rangle \cup \langle Int(Cl(Int(A_1))), \ldots, Int(Cl(Int(A_n)))\rangle\\
&\subseteq \langle A_1 \cup Int(Cl(Int(A_1))), \ldots, A_2 \cup Int(Cl(Int(A_n)))\rangle\\
&= \langle \beta Cl(A_1), \ldots, \beta Cl(A_n)\rangle
\end{aligned}$$

thereby completing the proof. □

Theorem 4.3. *Let $Y = \prod\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then*

$$Cl_{\theta_\beta}(\langle A_1, \cdots, A_n\rangle) \subseteq \langle Cl_{\theta_\beta}(A_1), \ldots, Cl_{\theta_\beta}(A_n)\rangle.$$

Proof. Let $x = \langle a_i\rangle \in Cl_{\theta_\beta}(\langle A_1, \cdots, A_n\rangle)$. Then for all open set U containing x, $\beta Cl(U) \cap \langle A_1, \cdots, A_n\rangle \neq \varnothing$. Suppose that for each j, there exists an open set U_j containing a_j such that $\beta Cl(U_j) \cap A_j = \varnothing$. Then $\langle U_1, \ldots, U_j, \ldots, U_n\rangle$ is an open set that contains x and by Theorem 4.2,

$$\begin{aligned}
&\beta Cl(\langle U_1, \ldots, U_j, \ldots, U_n\rangle) \cap \langle A_1, \ldots, A_j, \ldots, A_n\rangle\\
&\qquad \subseteq \langle \beta Cl(U_1) \cap A_1, \ldots, \beta Cl(U_j) \cap A_j, \ldots, \beta Cl(U_n) \cap A_n\rangle\\
&\qquad = \varnothing,
\end{aligned}$$

a contradiction. Therefore, $x \in \langle Cl_{\theta_\beta}(A_1), \ldots, Cl_{\theta_\beta}(A_n)\rangle$. □

Theorem 4.4. *Let $Y = \prod\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then*

$$\langle Int_{\theta_\beta}(A_1), \ldots, Int_{\theta_\beta}(A_n)\rangle \subseteq Int_{\theta_\beta}(\langle A_1, \cdots, A_n\rangle).$$

Proof. Let $x = \langle a_i\rangle \in \langle Int_{\theta_\beta}(A_1), \ldots, Int_{\theta_\beta}(A_n)\rangle$. Then $a_i \in Int_{\theta_\beta}(A_i)$ for all $i = 1, \ldots, n$. This means that there exists an open set U_i containing a_i such that $\beta Cl(U_i) \subseteq A_i$. Then $\langle U_1, \ldots, U_n\rangle$ is an open set containing x and so by Theorem 4.2

$$\beta Cl(\langle U_1, \ldots, U_n\rangle) \subseteq \langle \beta Cl(U_1), \ldots, \beta Cl(U_n)\rangle \subseteq \langle A_1, \cdots, A_n\rangle.$$

Thus, $x \in Int_{\theta_\beta}(\langle A_1, \cdots, A_n\rangle)$. □

Theorem 4.5. *Let $Y = \prod\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $\varnothing \neq O_i \subseteq Y_i$ for each $i = 1, \ldots, n$. If each O_i is θ_β-open in Y_i, then $O = \langle O_1, \ldots, O_n\rangle$ is θ_β-open in Y.*

Proof. Let $x = \langle a_i\rangle \in O$. Then $a_i \in O_i$ for all $i = 1, \ldots, n$. This implies that for each i, there exists an open set U_i containing a_i such that $\beta Cl(U_i) \subseteq O_i$. Let $U = \langle U_1, \ldots, U_n\rangle$. Then U is open containing x and by Theorem 4.2,

$$\begin{aligned}
\beta Cl(U) &= \beta Cl(\langle U_1, \ldots, U_n\rangle)\\
&\subseteq \langle \beta Cl(U_1), \ldots, \beta Cl(U_n)\rangle\\
&\subseteq \langle O_1, \ldots, O_n\rangle\\
&= O.
\end{aligned}$$

Hence, $O = \langle O_1, \ldots, O_n\rangle$ is θ_β-open in Y. □

Theorem 4.6. *Let $X = \prod\{X_i : 1 \leq i \leq n\}$ and $Y = \prod\{Y_i : 1 \leq i \leq n\}$ be finite product spaces and for each $i = 1, \dots, n$, let $f_i : X_i \to Y_i$ be a function. If each f_i is θ_β-continuous on X_i, then the function $f : X \to Y$ defined by $f(\langle x_i \rangle) = \langle f_i(x_i) \rangle$ is θ_β-continuous on X.*

Proof. Let $\langle V_1, \dots, V_n \rangle$ be a basic open set in Y. Then

$$f^{-1}(\langle V_1, \dots, V_n \rangle) = \langle f_1^{-1}(V_1), \dots, f_n^{-1}(V_n) \rangle.$$

Since each f_i is θ_β-continuous, $f_i^{-1}(V_i)$ is θ_β-open in X_i. Let $x = \langle x_i \rangle \in f^{-1}(\langle V_1, \dots, V_n \rangle)$. Then $x_i \in f_i^{-1}(V_i)$ for all $i = 1, \dots, n$. This means that there exists an open set O_i containing x_i such that $\beta Cl(O_i) \subseteq f_i^{-1}(V_i)$. Then $\langle O_1, \dots, O_n \rangle$ is open in X and contains x. By Theorem 4.2

$$\begin{aligned} \beta Cl(\langle O_1, \dots, O_n \rangle) &\subseteq \langle \beta Cl(O_1), \dots, \beta Cl(O_n) \rangle \\ &\subseteq \langle f_1^{-1}(V_1), \dots, f_n^{-1}(V_n) \rangle \\ &= f^{-1}(\langle V_1, \dots, V_n \rangle). \end{aligned}$$

This implies that $f^{-1}(\langle V_1, \dots, V_n \rangle)$ is θ_β-open on X. Therefore, f is θ_β-continuous on X. □

Theorem 4.7. *Let X be a topological space and $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space. A function $f : X \to Y$ is θ_β-continuous if and only if $p_\alpha \circ f$ is θ_β-continuous on X for every $\alpha \in \mathcal{A}$.*

Proof. Assume that f is θ_β-continuous on X. Let $a \in \mathcal{A}$ and U_α be open in Y_α. Since p_α is continuous, $p_\alpha^{-1}(U_\alpha)$ is open in Y. Hence,

$$f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$$

is θ_β-open in X. Therefore, $p_\alpha \circ f$ is θ_β-continuous on X for every $\alpha \in \mathcal{A}$.

Conversely, suppose that each coordinate function $p_\alpha \circ f$ is θ_β-continuous on X. Let $\langle O_\alpha \rangle$ be a subbasic open set in Y. Then O_α is open in Y_α for every $\alpha \in \mathcal{A}$ and

$$(p_\alpha \circ f)^{-1}(O_\alpha) = f^{-1}(p_\alpha^{-1}(O_\alpha) = f^{-1}(\langle O_\alpha \rangle)$$

is θ_β-open in X. Thus, f is θ_β-continuous on X. □

Corollary 4.8. *Let X be a topological space, $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space, and $f_\alpha : X \to Y_\alpha$ be a function for each $\alpha \in \mathcal{A}$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then f is θ_β-continuous on X if and only if each f_α is θ_β-continuous on X for each $\alpha \in \mathcal{A}$.*

Proof. For each $\alpha \in \mathcal{A}$ and every $x \in X$, we have

$$(p_\alpha \circ f)(x) = p_\alpha(f(x)) = p_\alpha(\langle f_\beta(x) \rangle) = f_\alpha(x).$$

Hence, $p_\alpha \circ f = f_\alpha$. The result follows from Theorem 4.7. □

5 θ_β-Connected Space and Versions of Separation Axioms

In this section, we provide characterizations of θ_β-connected space and some versions of separation axioms.

Definition 5.1. A topological space X is said to be a θ_β-*connected* if it is not the union of two nonempty disjoint θ_β-open sets. Otherwise, X is θ_β-*disconnected.*

Theorem 5.2. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_β-connected.*

(ii) *The only subsets of X that are both θ_β-open and θ_β-closed are $\varnothing$ and X.*

(iii) *No θ_β-continuous function $f : X \to \mathcal{D}$ is surjective.*

Proof. (i) $\Rightarrow$ (ii) Suppose that X is θ_β-connected. Let $F \subseteq X$ which is both θ_β-open and θ_β-closed. Then $X \setminus F$ is also both θ_β-open and θ_β-closed. Note that $X = F \cup (X \setminus F)$. Since X is θ_β-connected, either $F = \varnothing$ or $F = X$.

(ii) $\Rightarrow$ (iii) Suppose that (ii) holds and let $f : X \to \mathcal{D}$ be a θ_β-continuous surjection. Then $f^{-1}(\{0\}) \neq \varnothing, X$. Since $\{0\}$ is both open and closed in $\mathcal{D}$, $f^{-1}(\{0\})$ is both θ_β-open and θ_β-closed in X, a contradiction. Thus, (iii) follows.

(iii) $\Rightarrow$ (i) Assume that (iii) holds and let $X = A \cup B$, where A and B are nonempty disjoint θ_β-open sets. Then X is θ_β-disconnected. Note that A and B are also θ_β-closed sets. Consider the characteristic function $f_A : X \to \mathcal{D}$ of $A \subseteq X$, which is surjective. By Theorem 3.7, f_A is θ_β-continuous. This gives a contradiction. Thus, X must be θ_β-connected. $\square$

Theorem 5.3. Let X be a topological space. Then X is θ_β-connected if and only if X is θ-connected.

Proof. Assume that X is θ_β-connected. Then X cannot be the union of two nonempty disjoint θ_β-open sets. By Theorem 2.2 (i), every θ-open set is θ_β-open. It follows that X is not a union of θ-open sets. Accordingly, X is θ-connected.

Conversely, suppose that X is θ-connected. Then X is connected. Hence, X cannot be the union of two nonempty disjoint open sets. Since every θ_β-open set is open by Theorem 2.2 (ii), it follows that X is not the union of two nonempty disjoint θ_β-open sets. Therefore, X is θ_β-connected. $\square$

Corollary 5.4. *Let X be a topological space. Then X is θ_β-connected if and only if X is connected.*

Proof. Follows from Theorem 5.3 and from the fact that connected and θ-connected spaces are equivalent [21]. $\square$

Remark 5.5. The following diagram holds for a subset of a topological space.

$$\begin{array}{ccc} \beta\text{-connected} & \Longrightarrow & \text{connected} \\ & & \Updownarrow \\ \theta_\beta\text{-connected} & \Longleftrightarrow & \theta\text{-connected} \end{array}$$

The reverse implication for connected and β-connected spaces is not true as shown in the next example.

Example 5.6. Let $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Clearly, X is connected but not β-connected since $\{a, c\}$ and $\{b\}$ are two disjoint β-open sets, with $X = \{a, c\} \cup \{b\}$.

Definition 5.7. A topological space X is said to be

(i) *θ_β-Hausdorff* if given any pair of distinct points p, q in X, there exist disjoint θ_β-open sets U and V such that $p \in U$ and $q \in V$;

(ii) θ_β*-regular* if for each closed set F and each point $x \notin F$, there exist disjoint θ_β-open sets U and V such that $x \in U$ and $F \subseteq V$;

(iii) θ_β*-normal* if for every pair of disjoint closed sets E and F of X, there exist disjoint θ_β-open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Theorem 5.8. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_β-Hausdorff.*

(ii) *For distinct $x, w \in X$, there exists a θ_β-open set U containing x such that $w \notin Cl_{\theta_\beta}(U)$.*

(iii) *For each $x \in X$,*

$$C_x = \bigcap \{Cl_{\theta_\beta}(U) : U \text{ is } \theta_\beta\text{-open containing } x\} = \{x\}.$$

Proof. (i) $\Rightarrow$ (ii) Let X be θ_β-Hausdorff. By Definition 5.7 (i), for every pair of distinct points $x, w \in X$, there exist disjoint θ_β-open sets U and V such that $x \in U$ and $w \in V$. This means that $U \cap V = \varnothing$. Thus, $w \notin Cl_{\theta_\beta}(U)$.

(ii) $\Rightarrow$ (iii) Suppose that (ii) holds. Note that $x \in C_x$. By assumption, for every $x \neq w$, there exists a θ_β-open set U containing x such that $w \notin Cl_{\theta_\beta}(U)$. Thus, $w \notin C_x$. Since w is arbitrary, $C_x = \{x\}$.

(iii) $\Rightarrow$ (ii) Assume that (iii) holds. Let $x, w \in X$ such that $x \neq w$. By assumption, $x \in C_x$. Since $x \neq w$, $w \notin C_x$, that is, $w \notin \bigcap\{Cl_{\theta_\beta}(U) : U$ is θ_β-open containing $x\}$. This means that there exists a θ_β-open set U containing x such that $w \notin Cl_{\theta_\beta}(U)$. This completes the proof.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds. Let $x, w \in X$ such that $x \neq w$. By assumption, there exists a θ_β-open set U containing x such that $w \notin Cl_{\theta_\beta}(U)$. By Definition 2.13 (ii), there exists a θ_β-open set V containing w such that $U \cap V = \varnothing$. Hence, X is θ_β-Hausdorff. □

Theorem 5.9. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_β-regular.*

(ii) *For each $x \in X$ and an open set U containing x, there exists a θ_β-open set V such that $x \in V \subseteq Cl_{\theta_\beta}(V) \subseteq U$.*

(iii) *For each $x \in X$ and closed set F with $x \notin F$, there exists a θ_β-open set V containing x such that $F \cap Cl_{\theta_\beta}(V) = \varnothing$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that X is θ_β-regular. Let $x \in X$ and U be an open set containing x. Then $X \setminus U$ is closed and $x \notin X \setminus U$. By assumption, there exist disjoint open sets V and W such that $x \in V$ and $X \setminus U \subseteq W$. Since $V \cap W = \varnothing$, $V \subseteq X \setminus W$. By Theorem 2.14 (xii),

$$Cl_{\theta_\beta}(V) \subseteq Cl_{\theta_\beta}(X \setminus W) = X \setminus Int_{\theta_\beta}(W) = X \setminus W.$$

This means that $Cl_{\theta_\beta}(V) \cap W = \varnothing$. Consequently,

$$Cl_{\theta_\beta}(V) \cap (X \setminus U) \subseteq Cl_{\theta_\beta}(V) \cap W = \varnothing.$$

Hence, $Cl_{\theta_\beta}(V) \subseteq U$. Thus, $x \in V \subseteq Cl_{\theta_\beta}(V) \subseteq U$.

(ii) $\Rightarrow$ (iii) Suppose that (ii) holds. Let $x \in X$ and F be a closed set with $x \notin F$. Then $X \setminus F$ is open and $x \in X \setminus F$. By assumption, there exists a θ_β-open set V containing x such that $V \subseteq Cl_{\theta_\beta}(V) \subseteq X \setminus F$. This means that $F \cap Cl_{\theta_\beta}(V) = \varnothing$.

(iii) $\Rightarrow$ (i) Let $x \in X$ and F be a closed set such that $x \notin F$. By assumption, there exists a θ_β-open set V containing x such that $F \cap Cl_{\theta_\beta}(V) = \varnothing$. Observe that $X \setminus Cl_{\theta_\beta}(V)$ is a θ_β-open set and $F \subseteq X \setminus Cl_{\theta_\beta}(V)$. Since $V \subseteq Cl_{\theta_\beta}(V)$, $V \cap X \setminus Cl_{\theta_\beta}(V) = \varnothing$. Therefore, X is θ_β-regular. □

Theorem 5.10. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_β-normal.*

(ii) *For each closed set A and an open set $U \supseteq A$, there exists a θ_β-open set V containing A such that $Cl_{\theta_\beta}(V) \subseteq U$.*

(iii) *For each pair of disjoint closed sets A and B, there exists a θ_β-open set V containing A such that $Cl_{\theta_\beta}(V) \cap B = \varnothing$.*

Proof. (i) $\Rightarrow$ (ii) Assume that X is θ_β-normal. Let A be a closed set and U be an open set such that $A \subseteq U$. Then A and $X \setminus U$ are disjoint closed sets in X. By assumption, there exist disjoint θ_β-open sets V and W such that $A \subseteq V$ and $X \setminus U \subseteq W$. Since $X \setminus U \subseteq W$ and $V \cap W = \varnothing$, $X \setminus W \subseteq U$ and $V \subseteq X \setminus W$. By Theorem 2.14 (xii),

$$Cl_{\theta_\beta}(V) \subseteq Cl_{\theta_\beta}(X \setminus W) \subseteq X \setminus Int_{\theta_\beta}(W) = X \setminus W.$$

Thus, $Cl_{\theta_\beta}(V) \subseteq X \setminus W \subseteq U$.

(ii) $\Rightarrow$ (iii) Suppose that (ii) holds. Let A and B be a pair of disjoint closed sets in X. Then $A \subseteq X \setminus B$ and $X \setminus B$ is open. By assumption, there exists a θ_β-open set V containing A such that $Cl_{\theta_\beta}(V) \subseteq X \setminus B$. This means that $Cl_{\theta_\beta}(V) \cap B = \varnothing$.

(iii) $\Rightarrow$ (i) Suppose that (iii) holds. Let A and B be disjoint closed sets in X. By assumption, there exists a θ_β-open set V containing A such that $Cl_{\theta_\beta}(V) \cap B = \varnothing$. Then $B \subseteq X \setminus Cl_{\theta_\beta}(V)$. Observe that $Cl_{\theta_\beta}(V)$ is a θ_β-closed set. Thus, $X \setminus Cl_{\theta_\beta}(V)$ is a θ_β-open set. Since $V \subseteq Cl_{\theta_\beta}(V)$, $V \cap (X \setminus Cl_{\theta_\beta}(V)) = \varnothing$. Accordingly, X is θ_β-normal. □

A topological space X is said to be a T_1-space if for each $p, q \in X$ with $p \neq q$, there exist open sets U and V such that $p \in U, q \notin U$, and $q \in V, p \notin V$.

Theorem 5.11. *Let X be a T_1-space. Then the following statements hold:*

(i) *If X is θ_β-regular, then X is θ_β-Hausdorff.*

(ii) *If X is θ_β-normal, then X is θ_β-regular.*

Proof. (i) Assume that X is θ_β-regular. Let $x, w \in X$ with $x \neq w$. Since X is a T_1-space, there exist open sets U and V such that $x \in U$, $w \notin U$, and $w \in V$, $x \notin V$. This implies that $x \notin X \setminus U$, $w \in X \setminus U$, and $X \setminus U$ is closed. Since X is θ_β-regular, there exist disjoint θ_β-open sets A and B such that $x \in A$ and $X \setminus U \subseteq B$. Since $w \in X \setminus U$, $w \in B$. Thus, X is θ_β-Hausdorff.

(ii) Let X be θ_β-normal. Since X is a T_1-space, there exist open sets U and V such that $x \in U$, $w \notin U$, and $w \in V$, $x \notin V$. This implies that $x \notin X \setminus U$, $w \notin X \setminus V$ and $X \setminus U$ and $X \setminus V$ are disjoint closed sets. Since X is θ_β-normal, there exist disjoint θ_β-open sets E and F such that $X \setminus U \subseteq E$ and $X \setminus V \subseteq F$. Note that $x \in X \setminus V \subseteq F$. Hence $x \in F$ and $X \setminus U \subseteq E$. Therefore, X is θ_β-regular. □

By Theorem 5.11, we have the following remark.

Remark 5.12. For a T_1-space, the following diagram holds:

$$\theta_\beta\text{-normal} \Longrightarrow \theta_\beta\text{-regular} \Longrightarrow \theta_\beta\text{-Hausdorff}.$$

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