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Bernoulli Measure on the Sequence Space $\{-1, 1\}^{\mathbb{N}}$

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Abstract

This study develops a comprehensive measure-theoretic construction of a probability space on the sequence space $\Omega := \{-1, 1\}^{\mathbb{N}}$. By defining the cylinder σ-algebra on Ω and extending consistent finite-dimensional Bernoulli distributions through the Kolmogorov extension theorem, the study establishes the Bernoulli measure $\mu_{\mathfrak{B}}$ on the infinite-dimensional space Ω. The study then characterizes the Hilbert space $L^2(\mu_{\mathfrak{B}})$ by constructing an explicit orthonormal basis from normalized and centered coordinate projections and proving that $L^2(\mu_{\mathfrak{B}})$ is isomorphic to the space of square-summable sequences $\ell^2(\Gamma_0)$ in Γ_0, where Γ_0 is the collection of all finite subsets of the nonnegative numbers $\mathbb{N}_0$.

1 Introduction

In the 1970s, Takahiro Hida introduced the concept of white noise as a mathematical model of Gaussian randomness in infinite-dimensional spaces. His construction of the Gaussian white noise measure and its associated Hida test and distribution spaces provided a functional-analytic setting in which random functionals could be rigorously analyzed [4]. Subsequent developments refined this framework, leading to the formation of the Hida–Kondratiev spaces, which became central to stochastic calculus and infinite-dimensional analysis, see [5, 10, 11].

As the Gaussian framework matured, researchers began extending white noise analysis to non-Gaussian and discrete measures to describe jump-type or discontinuous stochastic behavior. Among the most prominent examples is the Poisson measure, which underlies the so-called Poisson white noise analysis [6, 9, 12]. Later, generalizations such as the fractional Poisson expanded the framework to accommodate a variety of non-Gaussian phenomena [1, 2, 13]. These constructions showed that the functional-analytic tools of Gaussian analysis, such as chaos expansions, Wick products, and distribution spaces, could be extended to non-Gaussian settings while preserving the infinite-dimensional structure.

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While these frameworks capture count-based randomness, they do not model binary structures that naturally arise in discrete-time stochastic systems. In this context, the Bernoulli measure assumes a fundamental role as the simplest discrete binary analog of the Gaussian measure. It serves as an essential intermediary between continuous Gaussian models and discrete Poisson-type frameworks. The rigorous construction of its infinite-dimensional version contributes to a more comprehensive foundation for discrete non-Gaussian analysis.

The Bernoulli distribution, first introduced by Jacob Bernoulli in his 1713 work "Ars Conjectandi", models random phenomena with two outcomes. Extending this distribution to an infinite sequence of independent Bernoulli trials gives rise to the Bernoulli measure on the sequence space $\{-1,1\}^{\mathbb{N}}$. This measure provides a natural setting for studying Bernoulli noise.

Several works have contributed to the development of Bernoulli noise and its analytical framework. Émery [3] investigated the chaotic representation property of discrete-time processes, including Bernoulli noise. Privault [14] formulated a discrete-time chaotic calculus, and Nourdin [15] applied Stein's method to the study of Rademacher functionals. Based on these ideas, Wang and Zhang [17] formulated a Wick-type analysis for Bernoulli noise functionals and introduced associated spaces of test and generalized functions that serve as an analog to the Hida triple in the Gaussian framework.

While analytic theories for Bernoulli noise exist, their measure-theoretic underpinnings remain largely implicit. This study addresses that gap by providing an explicit and systematic construction of the Bernoulli probability space, thereby making precise the foundational setting for infinite-dimensional Bernoulli analysis. The study begins with an exposition on the space of square-summable functions $\ell^2(\Gamma_0)$ on the finite configuration space Γ_0. It then defines a measurable structure on the sequence space $\{-1,1\}^{\mathbb{N}}$, establishes the Bernoulli measure by extending consistent finite-dimensional distributions, and characterizes the Hilbert space of square-integrable random variables associated with the infinite-dimensional Bernoulli probability space.

While the construction of the test and generalized function spaces of Bernoulli noise lies beyond the present scope, this work provides the essential groundwork for their future formulation. By bridging measure theory, probability, and infinite-dimensional analysis, the study contributes to the systematic extension of white noise analysis to discrete probability measures.

2 Finite Configuration Space

In this section, we review the finite configuration space Γ_0 and the sequence space $\ell^2(\Gamma_0)$. We then present an orthonormal basis for the space $\ell^2(\Gamma_0)$. Although the existence of this basis is well known in the literature [16, 17], proofs are often omitted. For completeness and to fix notation, we provide a self-contained exposition adapted to our setting. Throughout the paper, $\mathbb{N}_0$ denotes the set of nonnegative integers, while $\mathbb{N}$ denotes the set of all natural numbers.

Let $\Gamma_0 := \Gamma_{0,\mathbb{N}_0}$ be the finite configuration space on $\mathbb{N}_0$, that is

$$\Gamma_0 := \{\eta \subset \mathbb{N}_0 \mid |\eta| < \infty\},$$

where $|\eta|$ denotes the cardinality of the set η. The space Γ_0 is countable, and it is the disjoint union of the n-point configuration space $\Gamma^{(n)}_{0,\mathbb{N}_0}$, more precisely,

$$\Gamma_0 = \bigsqcup_{n\geq 0} \Gamma_0^{(n)}, \qquad \Gamma_0^{(n)} := \Gamma^{(n)}_{0,\mathbb{N}_0} := \{\eta \in \mathbb{N}_0 \mid |\eta| = n\}.$$

In addition, define the Hilbert space

$$\ell^2(\Gamma_0) = \left\{ f : \Gamma_0 \longrightarrow \mathbb{R} \mid \sum_{\eta\in\Gamma_0} |f(\eta)|^2 < \infty \right\}.$$

For $f, g \in \ell^2(\Gamma_0)$, the inner product is given by

$$\langle f, g\rangle_{\ell^2(\Gamma_0)} = \sum_{\eta\in\Gamma_0} f(\eta)g(\eta).$$

We denote the norm induced by the inner product on $\ell^2(\Gamma_0)$ by $\|\cdot\|_{\ell^2(\Gamma_0)}$. Moreover, we denote by $\ell^2(\mathbb{N}_0)$ the Hilbert space of square-summable real-valued sequences in $\mathbb{N}_0$, that is,

$$\ell^2(\mathbb{N}_0) = \left\{ x : \mathbb{N}_0 \longrightarrow \mathbb{R} \mid \sum_{n=0}^{\infty} |x_n|^2 < \infty \right\}.$$

For $x, y \in \ell^2(\mathbb{N}_0)$, the inner product is given by

$$\langle x, y\rangle_{\ell^2(\mathbb{N}_0)} = \sum_{n=0}^{\infty} x_n y_n.$$

For any sequence $x = (x_n)_{n\geq 0} \in \ell^2(\mathbb{N}_0)$, we define a real-valued function $\mathcal{E}_x : \Gamma_0 \longrightarrow \mathbb{R}$ by

$$\mathcal{E}_x(\eta) := \begin{cases} 1 & \text{if } \eta = \emptyset \\ \prod\limits_{k\in\eta} x_k, & \text{if } \eta \neq \emptyset. \end{cases}$$

Then $\mathcal{E}_x \in \ell^2(\Gamma_0)$ because

$$\|\mathcal{E}_x\|^2_{\ell^2(\Gamma_0)} = \sum_{\eta\in\Gamma_0} |\mathcal{E}_x(\eta)|^2 = \sum_{\eta\in\Gamma_0} \left| \prod_{k\in\eta} x_k \right|^2 = \sum_{\eta\in\Gamma_0} \prod_{k\in\eta} x_k^2 = \prod_{k=0}^{\infty} (1 + x_k^2)$$

and this product is finite if and only if $\sum\limits_{k=0}^{\infty} x_k^2 < \infty$, which is equivalent to saying $x \in \ell^2(\mathbb{N}_0)$. Let us find a natural basis for $\ell^2(\Gamma_0)$. Consider the following set of functions $S := \{\delta_\eta \mid \eta \in \Gamma_0\}$ where

$$\delta_\eta : \Gamma_0 \longrightarrow \mathbb{R},\ \xi \mapsto \delta_\eta(\xi) := \begin{cases} 1, & \text{if } \eta = \xi \\ 0, & \text{if } \eta \neq \xi. \end{cases}$$

Clearly, they are orthogonal since

$$\langle \delta_\eta, \delta_\xi\rangle_{\ell^2(\Gamma_0)} = \sum_{\theta\in\Gamma_0} \delta_\eta(\theta)\delta_\xi(\theta) = \delta_{\eta,\xi}$$

where $\delta_{\eta,\xi}$ is the Kronecker delta of η and ξ. Hence, if $f \in \ell^2(\Gamma_0)$, then

$$f = \sum_{\eta\in\Gamma_0} f(\eta)\delta_\eta \tag{1}$$

and for any $\xi \in \Gamma_0$, we have

$$\left(\sum_{\eta\in\Gamma_0} f(\eta)\delta_\eta \right)(\xi) = \sum_{\eta\in\Gamma_0} f(\eta)\delta_\eta(\xi) = f(\xi).$$

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As $\sum_{\eta\in\Gamma_0}|f(\eta)|^2<\infty$, the series in (1) is convergent since

$$\left\|\sum_{\eta\in\Gamma_0} f(\eta)\delta_\eta\right\|^2_{\ell^2(\Gamma_0)} = \sum_{\xi\in\Gamma_0}\left|\sum_{\eta\in\Gamma_0} f(\eta)\delta_\eta(\xi)\right|^2 = \sum_{\xi\in\Gamma_0}|f(\xi)^2|<\infty.$$

Note that the functions $\mathcal{E}_x$, $x\in\ell^2(\mathbb{N}_0)$, can be written on the basis S as follows

$$\mathcal{E}_x=\sum_{\eta\in\Gamma_0}\mathcal{E}_x(\eta)\delta_\eta.$$

3 Measurable Structure and Bernoulli Measure

In this section, we define a measurable structure on $\Omega=\{-1,1\}^{\mathbb{N}}$, using the coordinate projections and the cylinder σ-algebra. From the consistent family of probability distributions, we then obtain the Bernoulli measure $\mu_{\mathfrak{B}}$ via the Kolmogorov extension theorem.

Definition 3.1. Let Ω denote the space of all sequences with terms in $X=\{-1,1\}$, that is $\Omega=X^{\mathbb{N}}$. The elements of Ω are denoted by $\omega=(\omega_n)_{n\geq 1}\subset X$.

Observe that Ω is uncountable; this can be seen using the diagonal argument. On X, we consider the maximal σ-algebra of all its subsets

$$\mathscr{B}(X):=\mathscr{P}(\{-1,1\})=\big\{\emptyset,X,\{-1\},\{1\}\big\}.$$

Since X is finite, it is a Polish space with the discrete metric, and for any $n\in\mathbb{N}$, the Cartesian product X^n has the product σ-algebra which coincides with the σ-algebra $\mathscr{B}(X^n)$. More precisely, the generator $\mathcal{G}_n$ of the σ-algebra $\mathscr{B}(X^n)$ is the family of all sets of the form

$$B=B_1\times\cdots\times B_n,\ B_i\in\mathscr{B}(X),\ i=1,\ldots,n.$$

Hence, $\mathscr{B}(X^n):=\sigma(\mathcal{G}_n)$. It is clear that $\mathcal{G}_1=\mathscr{B}(X)$ is itself a σ-algebra.

Our objective is to detail the measurable structure on Ω, focusing on the σ-algebra of cylinder sets, and to establish the Bernoulli measure on Ω. We begin by establishing helpful notation to make the explanation of the necessary concepts later more straightforward.

Definition 3.2.

1. Let T be the set of finite sequences $\tilde{n}=(n_1,\ldots,n_k)$ of distinct natural numbers, where the length k of these sequences ranges over $\mathbb{N}$.
2. Suppose that for each $\tilde{n}\in T$ of length k, we have a probability measure $\mu_{\tilde{n}}$ on $\big(X^k,\mathscr{B}(X^k)\big)$. Then the collection, $\{\mu_{\tilde{n}},\ \tilde{n}\in T\}$, is called a family of finite-dimensional distributions.
3. The family $\{\mu_{\tilde{n}},\ \tilde{n}\in T\}$ is said to be *consistent*, provided that the following conditions are satisfied:
   (a) *Permutation invariance.* If $\tilde{m}=(n_{i_1},\ldots,n_{i_k})$ is a permutation of $\tilde{n}=(n_1,\ldots,n_k)$, then for any $B_i\in\mathscr{B}(X)$, $i=1,\ldots,k$, we have
   $$\mu_{\tilde{n}}(B_1\times\cdots\times B_k)=\mu_{\tilde{m}}(B_{i_1}\times\cdots\times B_{i_k}),\tag{C1}$$
   (b) *Projection invariance.* If $\tilde{n}=(n_1,\ldots,n_k)$ with $k\geq 1$, $\tilde{m}=(n_1,\ldots,n_{k-1})$, and $B\in\mathscr{B}(X^{k-1})$, then
   $$\mu_{\tilde{n}}(B\times X)=\mu_{\tilde{m}}(B).\tag{C2}$$

For any $k \in \mathbb{N}$ and $\tilde{n} = (n_1, \ldots, n_k) \in T$ define the projection from Ω onto X^k by

$$\pi_{\tilde{n}} : \Omega \longrightarrow X^k,\ \omega \mapsto \pi_{\tilde{n}}(\omega) := (w_{n_1}, \ldots, w_{n_k}).$$

Definition 3.3. Let $B \in \mathscr{B}(X^k)$ be given.

1. A k-dimensional cylinder set with base B on Ω is a set of the form
$$C_{\tilde{n},B} = \{\omega \in \Omega \mid (w_{n_1}, \ldots, w_{n_k}) \in B\} = \{\omega \in \Omega \mid \pi_{\tilde{n}}(\omega) \in B\} = \pi_{\tilde{n}}^{-1}(B). \tag{2}$$
2. We denote the algebra of all cylinder sets (of all finite dimensions) on Ω by $\mathscr{A}(\Omega)$ and by $\mathscr{C}(\Omega) := \sigma(\mathscr{A}(\Omega))$ the σ-algebra of cylinder sets generated by $\mathscr{A}(\Omega)$.

Example 3.4. Here we provide some examples of cylinder sets.

1. The set Ω is a cylinder set since $\Omega = \pi_1^{-1}(\{-1,1\}) \in \mathscr{A}(\Omega)$.
2. The set of all sequences that start with -1 is a cylinder set
$$\{\omega \in \Omega \mid w_1 = -1\} = \pi_1^{-1}(\{-1\}) \in \mathscr{A}(\Omega).$$
3. Let $M \subset \mathbb{N}$ be a finite set, say $M = \{n_1, \ldots, n_k\}$, then
$$C(M) := \bigcap_{i=1}^{k} \pi_{n_i}^{-1}(\{\omega_{n_i}\}) = \pi_{\tilde{n}}^{-1}\big(\{(\omega_{n_1}, \ldots, \omega_{n_k})\}\big) \in \mathscr{A}(\Omega),$$
where $\tilde{n} = (n_1, \ldots, n_k)$ and $\omega_{n_i} \in \{-1,1\}$.

Remark 3.5.
1. If we have a probability measure μ on $(\Omega, \mathscr{C}(\Omega))$, then we may define a family of finite-dimensional distributions by
$$\mu_{\tilde{n}}(B) := \mu\big(\{\omega \in \Omega \mid \pi_{\tilde{n}}(\omega) \in B\}\big), \tag{3}$$
for every $B \in \mathscr{B}(X^k)$ and all $\tilde{n} \in T$.
2. We are concerned with the reverse situation in which the probability measure derived on Ω becomes the Bernoulli measure.
3. For countable Ω, defining a probability measure on Ω is easier than in the uncountable case[1].

The desired result will follow from an application of the celebrated Kolmogorov extension theorem.

Theorem 3.6 (Daniel (1918), Kolmogorov (1933) [7])**.** *Let $\{\mu_{\tilde{n}} \mid \tilde{n} \in T\}$ be a consistent family of finite-dimensional distributions on $(X^k, \mathscr{B}(X^k))$. Then there is a probability measure μ on $(\Omega, \mathscr{C}(\Omega))$, such that (3) holds for every $\tilde{n} \in T$.*

[1]For a countable set $\Omega = \{\omega_1, \ldots, \omega_n, \ldots\}$ with the σ-algebra $\mathscr{P}(\Omega)$, following Kolmogorov [8], all probability measures on Ω appear as follows. Let $p_n \in [0,1]$ and $n \in \mathbb{N}$ be such that $p_1 + p_2 + \cdots = 1$; the sequence $(p_n)_{n\geq 1}$ is called a probability sequence. Any probability measure μ on Ω is defined by
$$\mu(A) = \sum_{n \mid \omega_n \in A} p_n = \sum_{n=1}^{\infty} p_n \delta_{w_n}(A),$$
where δ_ω is the Dirac measure at $\omega \in \Omega$.

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To illustrate the method for specifying the family of finite-dimensional distributions, we provide a simple example involving one-dimensional distributions.

Example 3.7. For any $0 < p < 1$, the one-dimensional marginal Bernoulli measure $\mu_{n,p}$, $n \in \mathbb{N}$, is the unique probability measure on $(X, \mathscr{B}(X))$ defined by

$$\mu_{n,p}(B) := p\delta_1(B) + (1-p)\delta_{-1}(B), \quad \forall B \in \mathscr{B}(X).$$

It is clear that $\mu_{n,p}$ is a probability measure on X.

With this example in mind, let us state the main result of this section.

Theorem 3.8. *Let $k \in \mathbb{N}$, $\tilde{n} \in T$, $0 < p_{n_i} < 1$, and $i = 1, \ldots, k$ be given. The family of probability measures $\{\mu_{\tilde{n},p} \mid \tilde{n} \in T\}$ on $(X^k, \mathscr{B}(X^k))$ defined by*

$$\mu_{\tilde{n},p}(B) := \prod_{i=1}^{k} \left[p_{n_i}\delta_1(B_i) + (1-p_{n_i})\delta_{-1}(B_i)\right], \quad B = B_1 \times \cdots \times B_k \in \mathscr{B}(X^k)$$

is a consistent family of finite-dimensional distributions. The probability measure on Ω arising from the family is called the Bernoulli measure and is denoted by $\mu_{\mathfrak{B}}$.

Proof. For any $\tilde{n} = (n_1, \ldots, n_k) \in T$, define the product measure

$$\mu_{\tilde{n},p} = \mu_{n_1,p_{n_1}} \otimes \cdots \otimes \mu_{n_k,p_{n_k}}$$

on $(X^k, \mathscr{B}(X^k))$ where each $\mu_{n_i,p_{n_i}}$ is a one-dimensional marginal Bernoulli measure on $(X, \mathscr{B}(X))$. Because each $\mu_{n_i,p_{n_i}}$ is a probability measure, the product measure exists and is unique. Moreover, for any measurable rectangle $B = B_1 \times \cdots \times B_k \in \mathscr{B}(X^k)$, we have

$$\mu_{\tilde{n},p}(B) = \prod_{i=1}^{k} \mu_{n_i,p_{n_i}}(B_i) = \prod_{i=1}^{k} \left[p_{n_i}\delta_1(B_i) + (1-p_{n_i})\delta_{-1}(B_i)\right]$$

Thus, for each $\tilde{n} \in T$, $\mu_{\tilde{n},p}$ is a well-defined probability measure on $(X^k, \mathscr{B}(X^k))$. It remains to show that this family of probability measures is consistent.

Claim 1: The family $\{\mu_{\tilde{n},p} \mid \tilde{n} \in T\}$ is permutation invariant.

Let $\tilde{m} = (n_{i_1}, \ldots, n_{i_k})$ be any permutation of $\tilde{n} = (n_1, \ldots, n_k) \in T$, and let $B_i \in \mathscr{B}(X)$, $i = 1, \ldots, k$. Since multiplication of real numbers is commutative, we have

$$\begin{aligned}
\mu_{\tilde{m},p}(B_{i_1} \times \cdots \times B_{i_k}) &= \mu_{n_{i_1},p_{n_{i_1}}}(B_{i_1}) \times \cdots \times \mu_{n_{i_k},p_{n_{i_k}}}(B_{i_k}) \\
&= \mu_{n_1,p_{n_1}}(B_1) \times \cdots \times \mu_{n_k,p_{n_k}}(B_k) \\
&= \prod_{i=1}^{k} \mu_{n_i,p_{n_i}}(B_i) \\
&= \mu_{\tilde{n},p}(B_1 \times \cdots \times B_k).
\end{aligned}$$

Hence, the family $\{\mu_{\tilde{n},p} \mid \tilde{n} \in T\}$ satisfies the permutation invariance property.

Claim 2: The family $\{\mu_{\tilde{n},p} \mid \tilde{n} \in T\}$ is projection invariant.

Let $\tilde{n} = (n_1, \ldots, n_k) \in T$, $\tilde{m} = (n_1, \ldots, n_{k-1})$, and $B = (B_1 \times \cdots \times B_{k-1}) \in \mathscr{B}(X^{k-1})$. Now,

$$\mu_{\tilde{n},p}(B \times X) = \left(\prod_{i=1}^{k-1} \mu_{n_i,p_{n_i}}(B_i)\right) \mu_{n_k,p_{n_k}}(X) = \prod_{i=1}^{k-1} \mu_{n_i,p_{n_i}}(B_i) \tag{4}$$

since $\mu_{n_k,p_{n_k}}(X)=1$. Note that $\prod_{i=1}^{k-1}\mu_{n_i,p_{n_i}}(B_i)=\mu_{\tilde{m},p}(B)$. Thus

$$\mu_{\tilde{n},p}(B\times X)=\mu_{\tilde{m},p}(B).$$

Accordingly, $\{\mu_{\tilde{n},p}\mid \tilde{n}\in T\}$ is a consistent family of finite-dimensional distributions. Moreover, the probability measure on Ω arising from the family is given by

$$\mu_{\mathfrak{B}}\circ\pi_{\tilde{n}}^{-1}(B)=\mu_{\tilde{n},p}(B)=\prod_{i=1}^{k}\left[p_{n_i}\delta_1(B_i)+(1-p_{n_i})\delta_{-1}(B_i)\right]$$

where $B=B_1\times\cdots\times B_k\in\mathscr{B}(X^k)$. □

The probability space $(\Omega,\mathscr{C}(\Omega),\mu_{\mathfrak{B}})$ is called the Bernoulli space.

4 Description of $L^2(\mu_{\mathfrak{B}})$

Now, we would like to describe the Hilbert space of measurable functions

$$L^2(\mu_{\mathfrak{B}}):=L^2(\Omega,\mathscr{C}(\Omega),\mu_{\mathfrak{B}})=\left\{F:(\Omega,\mathscr{C}(\Omega))\longrightarrow(\mathbb{R},\mathscr{B}(\mathbb{R}))\mid\int_\Omega|f(\omega)|^2\,\mathrm{d}\mu_{\mathfrak{B}}(\omega)<\infty\right\}$$

with inner product

$$(F,G)_{L^2(\mu_{\mathfrak{B}})}=\int_\Omega F(\omega)G(\omega)\,\mathrm{d}\mu_{\mathfrak{B}}(\omega).$$

We consider the family of projections $\{\pi_n\mid n\in\mathbb{N}\}$ from Ω onto X where

$$\pi_n:\Omega\longrightarrow X,\ \omega\mapsto\pi_n(\omega)=\omega_n,\quad n\in\mathbb{N}.$$

It is clear from the definition of the σ-algebra $\mathscr{C}(\Omega)$ that all projections π_n are $\mathscr{C}(\Omega)$-measurable. From now on, the expectation with respect to $\mu_{\mathfrak{B}}$ is denoted simply by $\mathbb{E}$.

Lemma 4.1. *The sequence of random variables $(\pi_n)_{n\in\mathbb{N}}$ has the following properties.*

1. *They are independent.*
2. *The expectation is given by $\mathbb{E}[\pi_n]=-1+2p_n$.*
3. *The variance of the centered random variable π_n is equal to $4p_n(1-p_n)$.*

Proof. 1. The independence of these random variables is a consequence of the definition of the measure on Ω. We provide the simple details for the reader convenience. We have to show that $\mathbb{E}[\pi_n\pi_m]=\mathbb{E}[\pi_n]\mathbb{E}[\pi_m]$. We have

$$\mathbb{E}[\pi_n\pi_m]=\int_\Omega\pi_n(\omega)\pi_m(\omega)\,\mathrm{d}\mu_{\mathfrak{B}}(\omega)=\int_{X^2}\omega_n\omega_m\,\mathrm{d}(\mu_{\mathfrak{B}}\circ(\pi_n^{-1},\pi_m^{-1}))(\omega_n,\omega_m).$$

Using the definition of the projection of the measure $\mu_{\mathfrak{B}}$, the integral on the right-hand side gives

$$\left(\int_X\omega_n\,\mathrm{d}(\mu_{\mathfrak{B}}\circ\pi_n^{-1})(\omega_n)\right)\left(\int_X\omega_m\,\mathrm{d}(\mu_{\mathfrak{B}}\circ\pi_m^{-1})(\omega_m)\right)=\mathbb{E}[\pi_n]\mathbb{E}[\pi_m].$$

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2. The expectation may be computed as follows

$$\begin{aligned}\mathbb{E}[\pi_n] &= \int_\Omega \pi_n(\omega)\,\mathrm{d}\mu_{\mathfrak{B}}(\omega)\\ &= \int_X \omega_n\,\mathrm{d}(\mu_{\mathfrak{B}}\circ\pi_n^{-1})(\omega_n)\\ &= \int_X \omega_n\,\mathrm{d}\big(p_n\delta_1(\omega_n)+(1-p_n)\delta_{-1}(\omega_n)\big)\\ &= p_n-(1-p_n) = -1+2p_n.\end{aligned}$$

3. We denote the centered sequence of random variables by $(\pi_{n,c} := \pi_n - (-1+2p_n))_{n\in\mathbb{N}}$. We have to compute $\mathbb{E}[\pi_{n,c}^2]$.

$$\begin{aligned}\mathbb{E}[\pi_{n,c}^2] &= -(-1+2p_n)^2 + \mathbb{E}_{\mu_{\mathfrak{B}}}[\pi_n^2]\\ &= -(-1+2p_n)^2 + \int_\Omega \pi_n^2(\omega)\,\mathrm{d}\mu_{\mathfrak{B}}(\omega)\\ &= -(-1+2p_n)^2 + \int_X \omega_n^2\,\mathrm{d}(\mu_{\mathfrak{B}}\circ\pi_n^{-1})(\omega_n)\\ &= -(-1+2p_n)^2 + \int_X \omega_n^2\,\mathrm{d}\big(p_n\delta_1(\omega_n)+(1-p_n)\delta_{-1}(\omega_n)\big)(\omega_n)\\ &= -(-1+2p_n)^2 + p_n + 1 - p_n\\ &= 4p_n(1-p_n).\end{aligned}$$

□

Hence, the sequence of random variables

$$Z_n := \frac{\pi_n+1-2p_n}{2\sqrt{p_n(1-p_n)}} = \frac{\pi_{n,c}}{\|\pi_{n,c}\|_{L^2(\mu_{\mathfrak{B}})}},\quad n\in\mathbb{N}.$$

is centered and normalized, that is, $\mathbb{E}[Z_n]=0$ and $\mathbb{E}[Z_n^2]=1$.

Proposition 4.2. *The sequence of random variables $(Z_n)_{n\geq 1}$ is independent and orthogonal in $L^2(\mu_{\mathfrak{B}})$.*

Proof. The independence of the Z_n's is a consequence of the fact that π_n's are independent. Now we show that $(Z_n, Z_m)_{L^2(\mu_{\mathfrak{B}})} = 0$ with $n \neq m$, which implies the orthogonality of Z_n's. We have,

$$\begin{aligned}(Z_n, Z_m)_{L^2(\mu_B)} &= \int_\Omega \frac{\pi_{n,c}(\omega)}{\|\pi_{n,c}\|_{L^2(\mu_{\mathfrak{B}})}}\cdot\frac{\pi_{m,c}(\omega)}{\|\pi_{m,c}\|_{L^2(\mu_{\mathfrak{B}})}}\,\mathrm{d}\mu_{\mathfrak{B}}(\omega)\\ &= \frac{1}{\|\pi_{n,c}\|_{L^2(\mu_{\mathfrak{B}})}}\cdot\frac{1}{\|\pi_{m,c}\|_{L^2(\mu_{\mathfrak{B}})}}\int_\Omega \pi_{n,c}(\omega)\pi_{m,c}(\omega)\,\mathrm{d}\mu_{\mathfrak{B}}(\omega)\\ &= \frac{1}{\|\pi_{n,c}\|_{L^2(\mu_{\mathfrak{B}})}\cdot\|\pi_{m,c}\|_{L^2(\mu_{\mathfrak{B}})}}\left(\int_\Omega \pi_n(\omega)\pi_m(\omega)\,\mathrm{d}\mu_{\mathfrak{B}}(\omega) - \mathbb{E}[\pi_n]\mathbb{E}[\pi_m]\right)\\ &= 0\end{aligned}$$

□

A cylinder function F on Ω is a function that can be written as

$$F(\omega) = g(\omega_{n_i}, \ldots, \omega_{n_k}),$$

where g is a continuous function of finitely many variables $\omega_{n_i}, \ldots, \omega_{n_k} \in X$. We denote the set of all cylinder functions by $\mathcal{FC}(\Omega)$.

Proposition 4.3. *The set of cylinder functions $\mathcal{FC}(\Omega)$ is dense in $L^2(\mu_{\mathfrak{B}})$.*

Proof. We have to show that every function in $L^2(\mu_{\mathfrak{B}})$ can be approximated by cylinder functions in the norm of $L^2(\mu_{\mathfrak{B}})$. In other words, for every function $F \in L^2(\mu_{\mathfrak{B}})$, there exists a sequence $(F_n)_{n\geq 1} \subset \mathcal{FC}(\Omega)$ such that

$$\lim_{n\to\infty} \|F - F_n\|_{L^2(\mu_{\mathfrak{B}})} = 0.$$

It follows from Theorem 1.9 in [5] that the algebra of cylinder polynomials is dense in L^p, and this implies the result via the Stone-Weierstrass theorem. □

Definition 4.4. For each $\eta \in \Gamma_0$ such that $\eta = \{n_1, \ldots, n_k\}$ define

$$Z_\eta := \prod_{i=1}^{k} Z_{n_i}$$

and $Z_\emptyset := 1$.

Theorem 4.5. *The family $(Z_\eta)_{\eta\in\Gamma_0}$ is a total orthonormal set in $L^2(\mu_{\mathfrak{B}})$.*

Proof. It is clear that for each $\eta = \{n_1, \ldots, n_k\} \subset \mathbb{N}$, $Z_\eta \in L^2(\mu_{\mathfrak{B}})$ because Z_{n_j}'s are independent (see Proposition 4.2) and normalized. As $L^2(\mu_{\mathfrak{B}})$ is a vector space, the subspace generated by Z_η's is included in $L^2(\mu_{\mathfrak{B}})$. It is clear that each Z_η, $\eta \in \Gamma_0$, is a cylinder function because

$$Z_\eta(\omega) = \prod_{i=1}^{k} \frac{\omega_{n_i} + 1 - 2p_{n_i}}{2\sqrt{p_{n_i}(1-p_{n_i})}} = g(w_{n_1}, \ldots, \omega_{n_k}),$$

where g is a continuous function. Moreover, every element in the subspace generated by the Z_η is a cylinder function. The claim follows from Proposition 4.3. □

Corollary 4.6. *Every function $F \in L^2(\mu_{\mathfrak{B}})$ can be written in terms of Z_η's; that is, there exist constants c_η such that*

$$F = \sum_{\eta\in\Gamma_0} F_\eta Z_\eta,$$

where the series converges in $L^2(\mu_{\mathfrak{B}})$, $F_\eta = (F, Z_\eta)_{L^2(\mu_{\mathfrak{B}})}$, and $\forall \eta \in \Gamma_0$.

Proposition 4.7. *The spaces $\ell^2(\Gamma_0)$ and $L^2(\mu_{\mathfrak{B}})$ are isomorphic.*

Proof. Define the linear map

$$\begin{aligned} J : \ell^2(\Gamma_0) &\longrightarrow L^2(\mu_B) \\ f &\mapsto J(f) = \sum_{\eta\in\Gamma_0} f(\eta) Z_\eta \end{aligned}$$

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where the series is taken as the limit (in $L^2(\mu_{\mathfrak{B}})$) of partial sums

$$S_N(f) = \sum_{k=1}^{N} f(\eta_k) Z_{\eta_k},$$

for an enumeration $(\eta_k)_{k\geq 1}$ of Γ_0. First, we show that J is well-defined. Let $f \in \ell^2(\Gamma_0)$ and let $\Gamma_0 = \{\eta_1, \eta_2, \eta_3, \dots\}$. For finite partial sums

$$S_N = \sum_{k=1}^{N} f(\eta_k) Z_{\eta_k},$$

and the fact that the family $(Z_\eta)_{\eta\in\Gamma_0}$ is orthonormal gives

$$\|S_N\|^2_{L^2(\mu_{\mathfrak{B}})} = \sum_{k=1}^{N} |f(\eta_k)|^2.$$

Take $M > N$. Now,

$$S_M(f) - S_N(f) = \sum_{k=N+1}^{M} f(\eta_k) Z_{\eta_k}$$

Thus,

$$\|S_M(f) - S_N(f)\|^2_{L^2(\mu_{\mathfrak{B}})} = \sum_{k=N+1}^{M} |f(\eta_k)|^2$$

Since $f \in \ell^2(\Gamma_0)$,

$$\sum_{k=N+1}^{\infty} |f(\eta_k)|^2 \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty$$

Hence,

$$\|S_M(f) - S_N(f)\|^2_{L^2(\mu_{\mathfrak{B}})} \longrightarrow 0 \quad \text{as} \quad M, N \longrightarrow \infty$$

Thus $(S_N(f))_{N\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space $L^2(\mu_{\mathfrak{B}})$, so it converges to some $F \in L^2(\mu_{\mathfrak{B}})$ which we denote by $J(f)$.

We now show that J is injective. For any $f \in \ell^2(\Gamma_0)$,

$$\|J(f)\|^2_{L^2(\mu_{\mathfrak{B}})} = \lim_{N\to\infty} \|S_N(f)\|^2 = \lim_{N\to\infty} \sum_{k=1}^{N} |f(\eta_k)|^2 = \sum_{\eta\in\Gamma_0} |f(\eta)|^2 = \|f\|^2_{\ell^2(\Gamma_0)}.$$

Thus, J is an isometry. The isometry of J implies that if $J(f) = 0$, then $f = 0$. Therefore, J is injective.

To show that J is surjective, let $F \in L^2(\mu_{\mathfrak{B}})$ be given. According to Corollary 4.6, F is represented by

$$F = \sum_{\eta\in\Gamma_0} F_\eta Z_\eta, \quad F_\eta \in \mathbb{R}, \ \forall \eta \in \Gamma_0.$$

Define $f_F : \Gamma_0 \longrightarrow \mathbb{R}$ by

$$f_F := \sum_{\eta\in\Gamma_0} F_\eta \delta_\eta.$$

We have $f_F \in \ell^2(\Gamma_0)$ due to the fact that $\{\delta_\eta \mid \eta \in \Gamma_0\}$ is an orthonormal basis of $\ell^2(\Gamma_0)$ and the fact that $F \in L^2(\mu_{\mathfrak{B}})$ implies that the series

$$\|f_F\|^2_{\ell^2(\Gamma_0)} = \sum_{\eta\in\Gamma_0} |F_\eta|^2.$$

is convergent. In addition, we have

$$J(f_F) = \sum_{\eta\in\Gamma_0} f_F(\eta) Z_\eta = \sum_{\eta\in\Gamma_0} \left(\sum_{\xi\in\Gamma_0} F_\xi \delta_\xi(\eta) \right) Z_\eta = \sum_{\eta\in\Gamma_0} F_\eta Z_\eta = F$$

showing that J is surjective. Accordingly, $\ell^2(\Gamma_0) \cong L^2(\mu_{\mathfrak{B}})$. □

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