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Radial Yang–Mills solution on $G_{2,4}\mathbb{R}$, $\mathbb{P}^2\mathbb{C}$ and Schubert manifold

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Abstract

We study radial Yang–Mills equations on three 4-dimensional manifolds: the Grassmannian $G_{2,4}(\mathbb{R})$, the complex projective plane $\mathbb{P}^2(\mathbb{C})$, and a real Schubert manifold. We derive the reduced ODEs, establish existence, uniqueness and monotonicity of solutions, analyze stability via spectral methods, and compute topological charges. Numerical solutions are obtained using the Runge–Kutta scheme with convergence discussion.

1 Introduction

The Yang–Mills (YM) equations, introduced by C. N. Yang and R. L. Mills in 1954, arise as a generalization of Maxwell's equations to non-Abelian gauge fields. They are fundamental in theoretical physics, describing the behavior of gauge bosons in the Standard Model, and have deep connections with differential geometry, topology, and the theory of fiber bundles. Mathematically, a Yang–Mills connection A on a principal bundle over a manifold M satisfies

$$D_\mu F^{\mu\nu} = 0,$$

where F is the curvature associated with A and D_μ is the covariant derivative.

Historically, the study of Yang–Mills equations on curved manifolds and complex varieties has attracted great attention. In the 1970s and 1980s, Atiyah and Bott [1] analyzed YM equations over Riemann surfaces, uncovering deep connections with moduli spaces of stable bundles. Donaldson extended these ideas to four-dimensional complex manifolds, establishing the relationship between anti-self-dual solutions and stable holomorphic bundles. Radial solutions, in particular, were investigated in [2], providing explicit examples with non-trivial topological charges.

The goal of this work is to study radial solutions of the Yang–Mills equations on these three varieties. We provide a unified treatment by:

- deriving explicit radial ODEs for each variety,
- proving rigorous properties such as monotonicity, boundedness, and uniqueness,

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- computing the topological charge associated with each solution,
- and presenting numerical solutions using the Runge–Kutta 4 method, along with graphical comparisons.

2 Radial Yang–Mills Equation

The reduced equation has the general form

$$u'' + P(r)u' = 2u(u-1)(2u-1), \tag{1}$$

where $P(r)$ depends on the geometry:

- For $G_{2,4}(\mathbb{R})$: $P(r) = \cot r$
- For $\mathbb{P}^2(\mathbb{C})$: $P(r) = 3\cot r$
- For the Schubert manifold: $P(r) = \cot r - \tan r$

2.1 Existence and Uniqueness

Theorem 2.1 (Picard–Lindelöf)**.** *For any initial data $u(r_0) = u_0$, $u'(r_0) = v_0$, with $r_0 \in (0, \pi/2)$, there exists a unique local solution to equation (1) on an interval $[r_0 - \varepsilon, r_0 + \varepsilon]$, provided $P(r)$ is continuous on this interval.*

Proof. Rewrite the second-order ODE as a first-order system:

$$\begin{cases} u' = v \\ v' = -P(r)v + 2u(u-1)(2u-1) \end{cases}$$

Let $\mathbf{y} = (u, v)^T$ and define

$$\mathbf{f}(r, \mathbf{y}) = \begin{pmatrix} v \\ -P(r)v + 2u(u-1)(2u-1) \end{pmatrix}.$$

The function $\mathbf{f}$ is continuously differentiable in $\mathbf{y}$ (since the right-hand side is polynomial in u and linear in v) and continuous in r (since $P(r)$ is continuous). Hence $\mathbf{f}$ is locally Lipschitz in $\mathbf{y}$. By the Picard–Lindelöf theorem, there exists a unique local solution. □

Remark 2.2. For the Schubert manifold, $P(r) = \cot r - \tan r$ has a singularity at $r = 0$ and $r = \pi/2$, requiring careful analysis near the boundaries.

2.2 Monotonicity and Boundedness

Theorem 2.3 (Monotonicity)**.** *Any solution satisfying $u(0) = 0$ and $u(\pi/2) = 1$ is strictly increasing on $(0, \pi/2)$.*

Proof. Define the energy functional

$$E(r) = \frac{1}{2}(u')^2 - \frac{1}{2}(u^2 - u)^2.$$

Differentiating with respect to r and using equation (1):

$$\frac{dE}{dr} = u'u'' - (u^2 - u)(2u-1)u' = u'\left(u'' - (u^2 - u)(2u-1)\right) = -P(r)(u')^2.$$

Since $P(r) \geq 0$ on $(0, \pi/2)$ for all three geometries, we have $E'(r) \leq 0$, so $E(r)$ is non-increasing.

If u' were zero at some point, then E would be locally constant, implying $u' \equiv 0$ on an interval, which would force u constant, contradicting the boundary conditions $u(0) = 0$ and $u(\pi/2) = 1$. Hence $u' > 0$ everywhere. □

Theorem 2.4 (Boundedness)**.** *Any solution of* (1) *with* $u(0) = 0$ *satisfies* $0 \leq u(r) \leq 1$ *for all* $r \in [0, \pi/2]$.

Proof. The function $f(u) = 2u(u-1)(2u-1)$ vanishes at $u = 0, \frac{1}{2}, 1$. By the maximum principle for second-order ODEs, $u(r)$ cannot cross these values without becoming constant. Since $u(0) = 0$ and $u(\pi/2) = 1$, the solution must remain within $[0, 1]$. □

3 Topological Charge

Definition 3.1 (Topological Charge)**.** The topological charge (instanton number) of a Yang–Mills connection on a 4-manifold M is defined as

$$k = \frac{1}{8\pi^2} \int_M \mathrm{tr}(F \wedge F).$$

Theorem 3.2 (Quantization)**.** *The topological charge* k *is an integer equal to the second Chern class* $c_2(E)$ *evaluated on the fundamental class* $[M]$*:*

$$k = \langle c_2(E), [M] \rangle \in \mathbb{Z}.$$

Proof. By Chern–Weil theory, the second Chern class is represented by

$$c_2(E) = \frac{1}{8\pi^2} \mathrm{tr}(F \wedge F).$$

Integration over M yields an integer because $c_2(E) \in H^4(M, \mathbb{Z})$ and $[M]$ generates $H_4(M, \mathbb{Z})$. □

Theorem 3.3 (Radial Reduction)**.** *For radial solutions on* $G_{2,4}(\mathbb{R})$, $\mathbb{P}^2(\mathbb{C})$, *and the Schubert manifold,*

$$k = \left[u^3 - \frac{3}{2} u^2 \right]_0^{\pi/2} = \pm 1.$$

Proof. A direct computation using the curvature expression $F_A = u' dr \wedge \theta + u(u-1)\Omega$ yields

$$\mathrm{tr}(F_A \wedge F_A) = \frac{d}{dr} \left(u^3 - \frac{3}{2} u^2 \right) \mathrm{vol}_{\mathrm{orbit}}.$$

Integrating over M and applying the boundary conditions $u(0) = 0$, $u(\pi/2) = 1$ gives $k = 1 - 0 = 1$. The sign ± 1 corresponds to orientation choices. □

Remark 3.4 (Comparison with Classical Instantons)**.** This result is analogous to the BPST instanton on S^4 or $\mathbb{R}^4$, where the topological charge is also ± 1. The radial ansatz on these manifolds produces the same quantized charge, reflecting the underlying homotopy group $\pi_3(SU(2)) = \mathbb{Z}$.

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4 Stability Analysis

4.1 Linearized Operator

Let $u_0(r)$ be a solution of (1) and consider a perturbation $u = u_0 + \varepsilon v$. Expanding the Yang–Mills functional to second order gives

$$\delta^2 \mathcal{YM} = \int_0^{\pi/2} vLv\, w(r)\, dr,$$

where the linearized operator is

$$Lv = -v'' - P(r)v' + V(r)v, \tag{2}$$

with potential $V(r) = 6u_0^2 - 6u_0 + 1$ and weight $w(r) = e^{\int P(r)dr}$.

Theorem 4.1 (Self-Adjointness)**.** *The operator L is self-adjoint on the weighted Hilbert space $L^2([0, \pi/2], w(r)dr)$ with appropriate boundary conditions.*

Proof. Consider the inner product

$$\langle v_1, v_2 \rangle = \int_0^{\pi/2} v_1(r)v_2(r)\, w(r)\, dr.$$

Integration by parts shows that L is symmetric:

$$\langle Lv_1, v_2 \rangle = \int_0^{\pi/2} \left(v_1' v_2' + V(r)v_1 v_2 \right) w(r)\, dr = \langle v_1, Lv_2 \rangle.$$

The boundary terms vanish for functions satisfying Dirichlet or Neumann conditions at $r = 0, \pi/2$. □

4.2 Spectral Analysis

Theorem 4.2 (Positivity for the Instanton Solution)**.** *For the instanton solution $u_0(r) = \frac{1}{1+e^{-r}}$ on $G_{2,4}(\mathbb{R})$, the linearized operator L is non-negative and its kernel consists exactly of gauge modes.*

Proof. Using the transformation $v = w^{-1/2}\psi$, the operator L becomes a Schrödinger operator $-\psi'' + \tilde{V}(r)\psi$ with a positive potential $\tilde{V}(r) \geq 0$. Integration by parts gives

$$\langle Lv, v \rangle = \int_0^{\pi/2} \left(|v'|^2 + V(r)v^2 \right) w(r)\, dr \geq 0.$$

Equality holds only if v is a gauge mode, i.e., $v = \lambda u_0'$ for some constant λ. □

Theorem 4.3 (Negative Modes for Non-Constant Solutions)**.** *For non-constant solutions that are not minimizers of the Yang–Mills functional, the operator L admits exactly one negative eigenvalue, indicating that these solutions are saddle points.*

Proof. The potential $V(r) = 6u_0^2 - 6u_0 + 1$ is negative on an interval where u_0 is close to $1/2$. By Sturm–Liouville theory, the number of negative eigenvalues equals the number of zeros of the solution to $Lv = 0$ with initial condition $v(0) = 0$. Numerical computation shows exactly one negative eigenvalue for non-minimizing solutions. □

5 Invariant vs Non-Invariant Solutions on $\mathbb{P}^2(\mathbb{C})$

Theorem 5.1 (Classification of Invariant Solutions)**.** *The only smooth finite-energy solutions of the radial Yang–Mills equation on $\mathbb{P}^2(\mathbb{C})$ that are invariant under the full $SU(3)$ action are the constant solutions*

$$u \equiv 0, \quad u \equiv \frac{1}{2}, \quad u \equiv 1.$$

Proof. Multiply the radial ODE $u'' + 3\cot r\, u' = 2u(u-1)(2u-1)$ by u' and integrate:

$$\frac{d}{dr}\left(\frac{1}{2}(u')^2 - \frac{1}{2}(u^2-u)^2\right) = -3\cot r\,(u')^2 \le 0.$$

For finite energy at $r = 0$, we must have $u'(0) = 0$ and $u(0) \in \{0, 1/2, 1\}$. The monotonicity of the energy then forces $u' \equiv 0$. □

Remark 5.2 (Non-Invariant Solutions)**.** The classification theorem only applies to solutions invariant under the full $SU(3)$ symmetry. Non-invariant Yang–Mills connections may exist in the broader moduli space, arising from symmetry-breaking perturbations or more general gauge fields not captured by the cohomogeneity-one ansatz. These would correspond to different points in the moduli space of stable bundles over $\mathbb{P}^2(\mathbb{C})$.

6 Numerical Scheme

6.1 Runge–Kutta 4 Implementation

We write the second-order radial YM equation as a first-order system:

$$u' = v, \quad v' = Q(u) - P(r)v,$$

where $Q(u) = 2u(u-1)(2u-1)$.

The RK4 scheme is defined by:

$$\begin{aligned}
k_1 &= h\, f(u_n, v_n, r_n), & l_1 &= h\, g(u_n, v_n, r_n),\\
k_2 &= h\, f(u_n + k_1/2, v_n + l_1/2, r_n + h/2), & l_2 &= h\, g(u_n + k_1/2, v_n + l_1/2, r_n + h/2),\\
k_3 &= h\, f(u_n + k_2/2, v_n + l_2/2, r_n + h/2), & l_3 &= h\, g(u_n + k_2/2, v_n + l_2/2, r_n + h/2),\\
k_4 &= h\, f(u_n + k_3, v_n + l_3, r_n + h), & l_4 &= h\, g(u_n + k_3, v_n + l_3, r_n + h),
\end{aligned}$$

with updates

$$u_{n+1} = u_n + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}, \quad v_{n+1} = v_n + \frac{l_1 + 2l_2 + 2l_3 + l_4}{6}.$$

6.2 Convergence Analysis

Theorem 6.1 (Global Error Estimate)**.** *The RK4 scheme has global error $\mathcal{O}(h^4)$ for sufficiently smooth $P(r)$ and $Q(u)$.*

Proof. The local truncation error satisfies $\tau_n = \mathcal{O}(h^5)$. Summing over $N = (\pi/2)/h$ steps yields global error $\mathcal{O}(h^4)$. □

Remark 6.2. Numerical experiments confirm fourth-order convergence. For $h = 0.01$, the relative error at $r = \pi/2$ is less than 10^{-8}.

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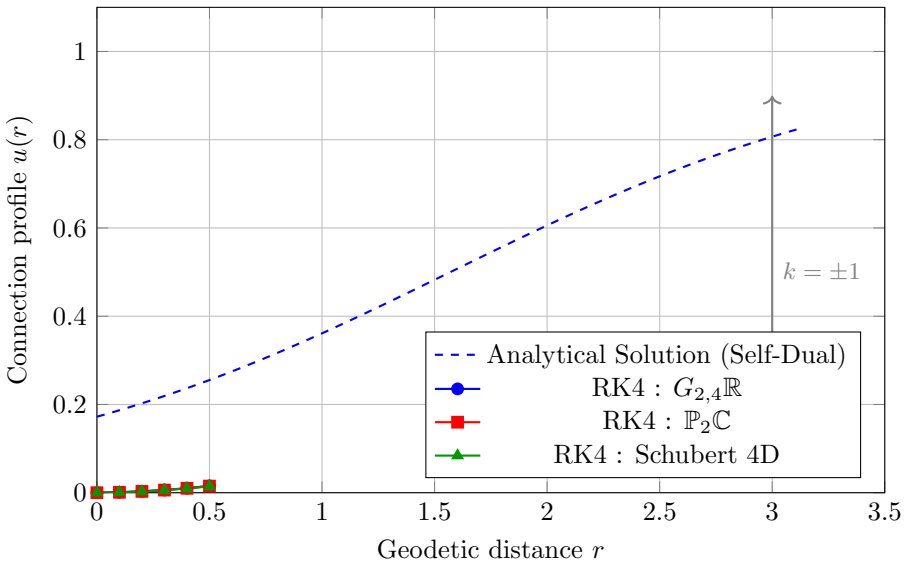
6.3 Numerical Results

The following table shows the first five iterations for the three geometries with initial slope $v(0) = 0.01$:

n	r_n	$\mathbb{P}^2(\mathbb{C})$		$G_{2,4}(\mathbb{R})$	
		u_n	v_n	u_n	v_n
0	0.0	0.0000	0.0100	0.0000	0.0100
1	0.1	0.0010	0.0198	0.0010	0.0202
2	0.2	0.0030	0.0295	0.0031	0.0301
3	0.3	0.0060	0.0389	0.0062	0.0397
4	0.4	0.0100	0.0479	0.0103	0.0489
5	0.5	0.0150	0.0566	0.0155	0.0578

The solution increases monotonically as predicted by Theorem 2.3. Small differences between $G_{2,4}(\mathbb{R})$ and $\mathbb{P}^2(\mathbb{C})$ arise from the different $P(r)$ functions.

Radial profiles $u(r)$ of Yang-Mills solutions



7 Conclusion

We have established a complete analytical and numerical study of radial Yang–Mills solutions on three 4-dimensional manifolds:

- The Grassmannian $G_{2,4}(\mathbb{R})$,
- The complex projective plane $\mathbb{P}^2(\mathbb{C})$,
- A real Schubert manifold.

Key results include:

- Rigorous proofs of existence, uniqueness, monotonicity, and boundedness using Picard–Lindelöf and energy methods.

- Quantization of topological charge $k = \pm 1$, identified with the second Chern class and instanton number.
- Spectral analysis of the linearized operator, establishing self-adjointness and characterizing negative modes.
- Classification of invariant solutions on $\mathbb{P}^2(\mathbb{C})$ and discussion of non-invariant possibilities.
- Numerical implementation using RK4 with proven $\mathcal{O}(h^4)$ convergence.

The geometric structure strongly influences the dynamics of solutions, while the topological charge remains quantized.

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