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Outer-interior Convex Roman Dominating Function in Graphs

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(*This paper is dedicated to my Professor Sergio R. Canoy Jr.*)

Abstract

Let $G = (V(G), E(G))$ be a connected graph and let $\rho : V(G) \to \{0, 1, 2\}$ be an RDF on G. For each $j \in \{0, 1, 2\}$, let $V_j = \{z \in V(G) : \rho(z) = j\}$. Then we write $\rho = (V_0, V_1, V_2)$. A function ρ is an outer-interior convex Roman dominating function (OInCvRDF) on G if (i) for each $v \in V_0$, there exists $u \in V_2$ such that $uv \in E(G)$, (ii) V_0 is an interior vertex set, and (iii) $V_1 \cup V_2$ is a convex set. The weight of OInCvRDF ρ, denoted by $\widetilde{\omega}_G^{InCvR}(\rho)$, is defined as $\widetilde{\omega}_G^{InCvR}(\rho) = \sum_{z \in V(G)} \rho(z)$. The outer-interior convex Roman domination number of G, denoted by $\widetilde{\gamma}_{InCvR}(G)$, is defined as $\widetilde{\gamma}_{InCvR}(G) = min\{\widetilde{\omega}_G^{InCvR}(\rho) : \rho \; is \; an$ OInCvRDF $on \; G\}$. Every OInCvRDF ρ on G with $\widetilde{\omega}_G^{InCvR}(\rho) = \widetilde{\gamma}_{InCvR}(G)$ is called a $\widetilde{\gamma}_{InCvR}$-function on G. In this paper, we introduce a new restricted version of convex Roman domination in graphs, called the outer-interior convex Roman dominating function, and present some of its graph-theoretic properties.

1 Introduction

One of the topics in graph theory that remains intriguing to many discrete mathematicians is the concept of a dominating set or domination in graphs. The concept of dominating set was pioneered by Claude Berge in 1958, and Oystein Ore in 1962 [1]. Domination in graphs has several applications in various areas of science, including resource allocation in economics, network system analysis in computer science, facility location problems in operations research, and many more. In addition, domination in graphs is very useful in protection strategies that involve placing guards at designated vertices to secure a graph (or network) against some attacks. Inspired by the defense strategy of the Roman Emperor Constantine the Great, Roman domination in graphs was invented by Cockayne et al. [3] in 2004. The model of Roman domination requires that every undefended location (no legions) is adjacent to a location with two legions, so that one legion could be transferred to protect the undefended location without leaving its original location unprotected, hence, the whole region is secured. In 2023, Fortosa

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and Canoy [5] introduced a new restricted variation of Roman domination in graphs called the convex Roman domination, which requires a convexity of the placement of the legions. With the idea of interior vertex set [2] on a graph and the convex Roman domination, the author is inspired to introduce a new parameter called the outer-interior convex Roman domination in graphs. The new parameter's feature is that the undefended region is adjacent to at least two locations, while the locations with legions are close to each other. This newly introduced concept can be linked to computer science, particularly in network analysis, and may contribute to the body of literature on domination in graphs.

2 Preliminaries

Let $G = (V(G), E(G))$ be a connected graph with $V(G)$ as the set of vertices and $E(G)$ as the set of edges of G. The cardinality of $V(G)$ is called the *order* of G, and the cardinality of $E(G)$ is called the *size* of G. The *open neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. Let $A \subseteq V(G)$. The set $N_G(A) = N(A) = \bigcup_{u \in A} N_G(u)$ is the *open neighborhood* of A and the set $N_G[A] = N[A] = N(A) \cup A$ is the *closed neighborhood* of A in G. Let $a, b \in V(G)$. The *distance* between a and b, denoted by $d_G(a, b)$, is the length of the shortest path between a and b. If there is no such walk (path) from a to b, then $d_G(a, b) = \infty$. Let $c \in V(G)$. If $d_G(a, b) = d_G(a, c) + d_G(c, b)$, then c is said to lie between a and b, and c is called an *interior vertex* of G [2]. Let $I \subseteq V(G)$. Then I is called an *interior vertex set* of G if for every $z \in I$, z is an interior vertex of G. An a-b path with $\infty > d_G(a, b) > 0$ is called an *a-b geodesic*. Let $I_G[a, b]$ be a set of closed intervals that consists of all vertices that lie on an a-b geodesic on G, and let $B \subseteq V(G)$. The union of all sets $I_G[a, b]$ where $a, b \in B$ is denoted by $I_G[B]$, and can be written as $\bigcup_{a,b \in B} I_G[a, b] = I_G[B]$. Then $d \in I_G[B]$ if and only if d is in some a-b geodesic for any $a, b \in B$. A set B is a *convex set* on G if and only if $I_G[B] = B$ [5]. The vertex set $V(G)$ is a convex set on G if the graph G is a connected graph.

A subset D_s of $V(G)$ is so-called a *dominating set* of G if for each $v \in V(G) \setminus D_s$, there exists $u \in D_s$ such that $v \in N_G(u)$ [7]. The *domination number* of G, denoted by $\gamma(G)$, is defined as the smallest cardinality of D_s on G. If D_s satisfies the condition $|D_s| = \gamma(G)$, then D_s is called the γ-set on G. A subset C_s of $V(G)$ is so-called a *convex dominating set* of G if for each $v \in V(G) \setminus C_s$, there exists $u \in C_s$ such that $v \in N_G(u)$ and C_s is a convex set on G [4]. The *convex domination number* of G, denoted by $\gamma_{Cv}(G)$, is defined as the smallest cardinality of C_s on G. If C_s satisfies the condition $|C_s| = \gamma_{Cv}(G)$, then C_s is called the γ_{Cv}-set on G. A subset I_o of $V(G)$ is so-called an *outer-interior dominating set* of G if for each $v \in V(G) \setminus I_o$, there exists $u \in I_o$ such that $uv \in E(G)$ and $V(G) \setminus I_o$ is an interior vertex set on G. The *outer-interior domination number* of G, denoted by $\widetilde{\gamma}_{In}(G)$, is defined as the minimum cardinality of I_o on G. If I satisfies the condition $|I_o| = \widetilde{\gamma}_{In}(G)$, then I_o is called the $\widetilde{\gamma}_{In}$-set on G. A subset I_c of $V(G)$ is so-called an *outer-interior convex dominating set* of G if for each $v \in V(G) \setminus I_c$, there exists $u \in I_c$ such that $v \in N_G(u)$, I_c is convex, and $V(G) \setminus I_c$ is an interior vertex set on G. The *outer-interior convex domination number* of G, denoted by $\widetilde{\gamma}_{InCv}(G)$, is defined as the smallest cardinality of I_c on G. If I_c satisfies the condition $|I_c| = \widetilde{\gamma}_{InCv}(G)$, then I_c is called the $\widetilde{\gamma}_{InCv}$-set on G.

Let $\rho : V(G) \to \{0, 1, 2\}$ be a function on G. Then consider the following: $V_0 = \{x \in V(G) : \rho(x) = 0\}$, $V_1 = \{x \in V(G) : \rho(x) = 1\}$, and $V_2 = \{x \in V(G) : \rho(x) = 2\}$. In that case, we write $\rho = (V_0, V_1, V_2)$. A function ρ is called a *Roman dominating function* (RDF) if for each $v \in V_0$, there exists $u \in V_2$ such that $v \in N_G(u)$ [3]. The *weight* of a function ρ is given by $\omega_G^R(\rho) = \sum_{x \in V(G)} \rho(x) = |V_1| + 2|V_2|$. The smallest weight of an RDF ρ on G is called a *Roman domination number* of G and is denoted by $\gamma_R(G)$, that is, $\gamma_R(G) =$

$min\{\omega_G^R(\rho) : \rho\ is\ an$ RDF $on\ G\}$. Every RDF ρ on G satisfying the condition $\omega_G^R(\rho) = \gamma_R(G)$ is called a γ_R-function on G. A function ρ is called a convex Roman dominating function (CvRDF) on G if for each $v \in V_0$, there exists $u \in V_2$ such that $v \in N_G(u)$, and $V_1 \cup V_2$ is a convex set on G [5]. The weight of CvRDF ρ is denoted by $\omega_G^{CvR}(\rho)$, and is defined as $\omega_G^{CvR}(\rho) = \sum_{v \in V(G)} \rho(v) = |V_1| + 2|V_2|$. The convex Roman domination number of G is denoted by $\gamma_{CvR}(G)$, and is defined as the smallest weight of an CvRDF ρ on G, that is, $\gamma_{CvR}(G) = min\{\omega_G^{CvR}(\rho) : \rho\ is\ an$ CvRDF $on\ G\}$. Every CvRDF ρ on G satisfying the condition $\omega_G^{CvR}(\rho) = \gamma_{CvR}(G)$ is called a γ_{CvR}-function on G. A function ρ is called an outer-interior convex Roman dominating function (OInCvRDF) on G if (i) for each $v \in V_0$, there exists $u \in V_2$ such that $v \in N_G(u)$, (ii) $V_1 \cup V_2$ is a convex set, and (iii) V_0 is an interior vertex set on G. The weight of OInCvRDF ρ is denoted by $\widetilde{\omega}_G^{InCvR}(\rho)$, and is defined as $\widetilde{\omega}_G^{InCvR}(\rho) = \sum_{x \in V(G)} \rho(x) = |V_1| + 2|V_2|$. The outer-interior convex Roman domination number of G is denoted by $\widetilde{\gamma}_{InCvR}(G)$, and is defined as the smallest weight of an OInCvRDF ρ on G, that is, $\widetilde{\gamma}_{InCvR}(G) = min\{\widetilde{\omega}_G^{InCvR}(\rho) : \rho\ is\ an$ OInCvRDF $on\ G\}$. Every OInCvRDF ρ on G that satisfies the condition $\widetilde{\omega}_G^{InCvR}(\rho) = \widetilde{\gamma}_{InCvR}(G)$ is called a $\widetilde{\gamma}_{InCvR}$-function on G.

Consider the graph G of order $n = 11$ in the Figure 1. Let $\rho = (V_0, V_1, V_2)$ be an RDF on G such that $V_0 = \{v_2, v_3, v_4, v_7, v_8, v_9, v_{10}, v_{11}\}$, $V_1 = \{v_1\}$, and $V_2 = \{v_5, v_6\}$. Then it is easy to see that ρ is a $\widetilde{\gamma}_{InCvR}$-function on G. Thus, we obtain $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = |\{v_1\}| + 2|\{v_5, v_6\}| = 5$.

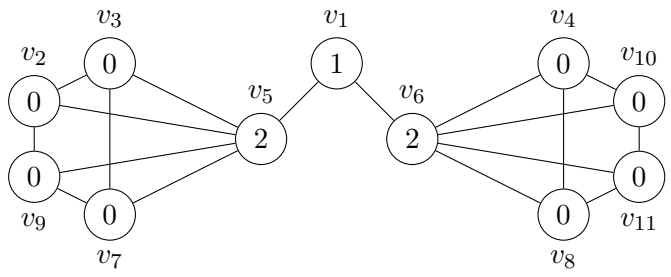


Figure 1: A graph G with $|V(G)| = 11$ and $\widetilde{\gamma}_{InCvR}(G) = 5$

For the terminologies and concepts used in this study which are not introduced here, the readers are advised to refer to [6, 7]. In this paper, a new variant of convex Roman domination called the outer-interior convex Roman domination in graphs is introduced and investigated. This paper explored the combinatorial properties of outer-interior convex Roman domination in some graph classes, characterizations were constructed, and obtained some bounds.

3 Results

This section presents some graph-theoretic properties of the newly introduced parameter called outer-interior convex Roman dominating function in graphs. Due to the convexity of the dominating set, this research considers connected graphs only. We start with a remark as a quick consequence of the definition of OInCvRDF.

Remark 3.1. Let G be a connected graph of order n and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. If $\widetilde{\gamma}_{InCvR}(G) < n$, then $V_0 \neq \varnothing$ is an interior vertex set on G and $V_1 \cup V_2 = I_G[V_1 \cup V_2]$.

The following are some important properties of OInCvRDF.

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Proposition 3.2. *Let G be a connected graph. If $\rho = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{InCvR}$-function on G, then $V_1 \cup V_2$ is an outer-interior convex dominating set on G.*

Proof. Assume that $\rho = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{InCvR}$-function on G. Then it means that ρ is an OInCvRDF on G. Thus, it also implies that ρ is an RDF on G. Consequently, it follows that $V_1 \cup V_2$ is a dominating set on G. Let $x \in V_0$. By definition of OInCvRDF, there exists $y \in V_2$ such that $xy \in E(G)$, V_0 is an interior vertex set on G, and $V_1 \cup V_2$ is convex. Now, since $V_0 = V(G) \setminus (V_1 \cup V_2)$ and $V_1 \cup V_2$ is a dominating set on G, we have that $V_1 \cup V_2$ is an outer-interior convex dominating set on G. This proves the assertion. □

Corollary 3.3. *Let G be a connected graph. If $\rho = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{InCvR}$-function on G, then $V_1 \cup V_2$ is not always a $\widetilde{\gamma}_{InCv}$-set on G.*

Proof. Let $G = C_4 = [p_1, p_2, p_3, p_4, p_1, ...]$ and let $\rho = (V_0, V_1, V_2)$ be an RDF on G. Let $V_0 = \{p_1\}$, $V_1 = \{p_3, p_4\}$ and $V_2 = \{p_2\}$. Then V_0 is an interior vertex set and $V_1 \cup V_2$ is convex. Clearly, $\rho = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{InCvR}$-function on G. Let O_i be an outer-interior convex dominating set. Then it is easy to see that $O_i = \{p_2, p_3\}$ and $\{p_1, p_4\} = V(G) \setminus O_i$ is an interior vertex set. Accordingly, $V_1 \cup V_2$ is not a $\widetilde{\gamma}_{InCv}$-set on G. □

If we set $|V_1| = 0$, then we have the following result.

Theorem 3.4. *Let G be a connected graph and let $\rho = (V_0, V_1, V_2)$ be an OInCvRDF on G for which $|V_1| = 0$. Then ρ is $\widetilde{\gamma}_{InCvR}$-function on G if and only if V_2 is a $\widetilde{\gamma}_{InCv}$-set on G.*

Proof. Let $\rho = (V_0, V_1 = \varnothing, V_2)$ be an OInCvRDF on G. Assume that ρ is $\widetilde{\gamma}_{InCvR}$-function on G. Suppose, for the contrary that, V_2 is not a $\widetilde{\gamma}_{InCv}$-set on G. This implies that there exists $\widetilde{\gamma}_{InCv}$-set U_2 on G such that $V(G) \setminus U_2$ is an interior vertex set on G and U_2 is a convex set. Let $W_0 = V(G) \setminus U_2$, $W_1 = \varnothing$, and $W_2 = U_2$. Then $\rho' = (W_0, W_1, W_2)$ is an OInCvRDF on G. In that case, we get $\widetilde{\omega}_G^{InCvR}(\rho') = |W_1| + 2|W_2| = 2|U_2| < 2|V_2| = |V_1| + 2|V_2| = \widetilde{\omega}_G^{InCvR}(\rho) = \widetilde{\gamma}_{InCvR}(G)$. This is a contradiction. Therefore, it suffices to say that V_2 is a $\widetilde{\gamma}_{InCv}$-set on G. Assume that V_2 is a $\widetilde{\gamma}_{InCv}$-set on G. Then it follows that $V_0 = V(G) \setminus V_2$ is an interior vertex set on G and V_2 is a convex set. Suppose, for the contrary that, $\rho = (V_0, V_1 = \varnothing, V_2)$ is an OInCvRDF on G but not a $\widetilde{\gamma}_{InCvR}$-function on G. Then there exists $\widetilde{\gamma}_{InCvR}$-function $\rho'' = (V_0', V_1' = \varnothing, V_2')$ on G. So, we have $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho'') = |V_1'| + 2|V_2'| = 2|V_2'| < 2|V_2| = |V_1| + 2|V_2| = \widetilde{\omega}_G^{InCvR}(\rho)$. Thus, $|V_2'| < |V_2|$, a contradiction. Therefore, $\rho = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{InCvR}$-function on G. This completes the proof. □

Theorem 3.5. *Let G be a connected graph and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Then $|V_0| = 0$ if and only if $|V_2| = 0$.*

Proof. Let $\rho = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{InCvR}$-function on G. Then ρ is an OInCvRDF on G. Suppose $|V_0| = 0$. Assume for a moment that $|V_2| > 0$. Let $x \in V_2$ and let $V_0' = V_0$, $V_1' = V_1 \cup \{x\}$, and $V_2' = V_2 \setminus \{x\}$. Then it follows that $\rho' = (V_0', V_1', V_2')$ is an outer-interior convex Roman dominating function on G. Thus, we have that $\widetilde{\omega}_G^{InCvR}(\rho') = |V_1'| + 2|V_2'| = (|V_1| + 1) + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 < \widetilde{\omega}_G^{InCvR}(\rho) = \widetilde{\gamma}_{InCvR}(G)$. This is a contradiction. Therefore, we conclude that $|V_2| = 0$. Conversely, we suppose that $|V_2| = 0$. Since ρ is an OInCvRDF on G, it is forced that $|V_0| = 0$. This proves the assertion. □

Remark 3.6. Let G be a connected graph of order n and let $\rho = (V_0, V_1, V_2)$ be an OInCvRDF on G. If $|V_0| = 0 = |V_2|$, then $\widetilde{\gamma}_{InCvR}(G) = n$.

Theorem 3.7. *Let G be a connected graph of order n. Then*

$$max\{\gamma_R(G), \widetilde{\gamma}_{InCv}(G)\} \leq \widetilde{\gamma}_{InCvR}(G) \leq min\{2\widetilde{\gamma}_{InCv}(G), n\}.$$

Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. By Theorem 3.2, $V_1 \cup V_2$ is an outer-interior convex dominating set on G. In that case, we have that $\widetilde{\gamma}_{InCv}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \widetilde{\gamma}_{InCvR}(G)$. Moreover, since every OInCvRDF on G is an RDF on G, it means that $\gamma_R(G) \leq \widetilde{\gamma}_{InCvR}(G)$. Thus, $max\{\widetilde{\gamma}_{InCv}(G), \gamma_R(G)\} \leq \widetilde{\gamma}_{InCvR}(G)$. Now, let $|V_0| = 0$. Then by Theorem 3.5, $|V_2| = 0$. Hence, $V_1 = V(G)$. Since $\rho = (\varnothing, V_1 = V(G), \varnothing)$ is an OInCvRDF on G, it follows that $\widetilde{\gamma}_{InCvR}(G) \leq \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. Thus, $\widetilde{\gamma}_{InCvR}(G) \leq n$. Let O be a $\widetilde{\gamma}_{InCv}$-set on G. Then $\widetilde{\gamma}_{InCv}(G) = |O|$. Let $W_0 = V(G) \setminus O$, $W_1 = \varnothing$, and $W_2 = O$. Then $\rho^* = (W_0, W_1, W_2)$ is an OInCvRDF on G, that is, $W_0 = V(G) \setminus O$ is an interior vertex set on G and W_2 is convex set. So, we have $\widetilde{\gamma}_{InCvR}(G) \leq \widetilde{\omega}_G^{InCvR}(\rho^*) = |W_1| + 2|W_2| = 2|O| = 2\widetilde{\gamma}_{InCv}(G)$. Consequently, we obtain $\widetilde{\gamma}_{InCvR}(G) \leq min\{n, 2\widetilde{\gamma}_{InCv}(G)\}$. This completes the proof. □

Theorem 3.8. *Let G be a connected graph of order n and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Then $\widetilde{\gamma}_{InCvR}(G) = n$ if and only if $|V_2| = |V_0| \geq 0$.*

Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Assume that $\widetilde{\gamma}_{ICvR}(G) = n$. Suppose, for the contrary, that $|V_2| \neq |V_0|$. Then either $|V_2| < |V_0|$ or $|V_2| > |V_0|$. If $|V_2| < |V_0|$, then we have $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| < |V_0| + |V_1| + |V_2| = |V(G)| = n$. Now, if $|V_2| > |V_0|$, then we obtain $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| > |V_0| + |V_1| + |V_2| = |V(G)| = n$. Hence, if $|V_2| \neq |V_0|$, then $\widetilde{\gamma}_{InCvR}(G) \neq n$. This is a contradiction. Therefore, we conclude that $|V_2| = |V_0|$. Conversely, assume that $|V_2| = |V_0| \geq 0$. On one hand, if $|V_2| = |V_0| = 0$, then by Remark 3.6, $\widetilde{\gamma}_{InCvR}(G) = n$. On the other hand, if $|V_2| = |V_0| \geq 1$, then $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = |V_0| + |V_1| + |V_2| = |V(G)| = n$. This completes the proof. □

The next corollary is an immediate consequence of Theorem 3.8.

Corollary 3.9. *Let G be a connected graph of order n and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. If $|V_0| = 1$, then $\widetilde{\gamma}_{InCvR}(G) = n$.*

Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Then ρ is an OInCvRDF on G. Assume that $|V_0| = 1$. Since ρ is an OInCvRDF on G, it follows that $V_2 \neq \varnothing$. If $|V_2| > 1$, then $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| > |V_0| + |V_1| + |V_2| = |V| = n$. This is a contradiction to Theorem 3.7. Hence, we have that $|V_2| = 1$. By Theorem 3.8, it suffices to conclude that $\widetilde{\gamma}_{InCvR}(G) = n$. □

Theorem 3.10. *If $G = K_n$ (complete graph) where $n \in \mathbb{N}$, then $\widetilde{\gamma}_{InCvR}(G) = n$.*

Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Let $n = 1$. Then $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| = |V(K_1)| = 1$. Let $n = 2$. Then it follows that $|V_2| = |V_0| = 1$ and $V_1 = \varnothing$. Thus, we have that $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = 2|V_2| = 2$. Let $n \geq 3$. Since for every $x \in V(G)$, there does not exist $a, b \in V(G) \setminus \{x\}$ such that $d_G(a, b) = d_G(a, x) + d_G(x, b)$, it follows that $|V_0| = 0$. By Theorem 3.5, $|V_2| = 0$. By Remark 3.6, it is concluded that $\widetilde{\gamma}_{InCvR}(G) = n$. This proves the assertion. □

Corollary 3.11. *Let G and H be complete graphs of order n and m, respectively. Then $\widetilde{\gamma}_{InCvR}(G + H) = n + m$.*

Theorem 3.12. *Let G be a connected graph of order n and let $\rho = (V_0, V_1, V_2)$ be $\widetilde{\gamma}_{InCvR}$-function on G. Then $\widetilde{\gamma}_{InCvR}(G) < n$ if and only if $1 \leq |V_2| < |V_0|$.*

Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Assume that $\widetilde{\gamma}_{InCvR}(G) < n$. Then $\widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| < |V(G)| = |V_0| + |V_1| + |V_2|$. Hence, we have that $1 \leq |V_2| < |V_0|$. Conversely, we assume that $1 \leq |V_2| < |V_0|$. Suppose, for the contrary that, $\widetilde{\gamma}_{InCvR}(G) = n$.

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By Theorem 3.8, $|V_0| = |V_2| \geq 0$. This is a contradiction. Therefore, it suffices to say that $\widetilde{\gamma}_{InCvR}(G) < n$. This proves the assertion. □

Corollary 3.13. *Let G be a connected graph of order n. If $\rho = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{InCvR}$-function on G, then $0 \leq |V_2| \leq |V_0|$.*

Proof. Assume that $\rho = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{InCvR}$-function on G. By Theorem 3.7, $\widetilde{\gamma}_{InCvR}(G) \leq n$. Let $\widetilde{\gamma}_{InCvR}(G) = n$. By Theorem 3.8, $|V_2| = |V_0| \geq 0$. Let $\widetilde{\gamma}_{InCvR}(G) < n$. By Theorem 3.12, $1 \leq |V_2| < |V_0|$. Therefore, $0 \leq |V_2| \leq |V_0|$. This proves the assertion. □

Theorem 3.14. *Let G be a connected graph. Then*

(*i*) $\widetilde{\gamma}_{InCvR}(G) = 1$ *if and only if* $G = K_1$; *and*

(*ii*) $\widetilde{\gamma}_{InCvR}(G) = 2$ *if and only if* $G \in \{K_2, K_1 + H\}$ *where* $V(H)$ *is an interior vertex set on* G.

Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Then ρ is an OInCvRDF on G. Assume that $\widetilde{\gamma}_{InCvR}(G) = 1$. Suppose, for the contrary that, $G \neq K_1$. Then $|V(G)| > 1$. Let $|V_0| \neq 0$. Then by Theorem 3.5, $|V_2| \neq 0$. Hence, we obtain $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| > 1$, a contradiction. Let $|V_0| = 0$. Then by Theorem 3.5, $|V_2| = 0$. This implies that $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = |V_1| = |V(G)| > 1$, a contradiction. Therefore, it suffices to say that $G = K_1$. Conversely, assume that $G = K_1$. By Theorem 3.10, $\widetilde{\gamma}_{InCvR}(G) = 1$ and so, (*i*) is satisfied. Assume that $\widetilde{\gamma}_{InCvR}(G) = 2$. Then $\widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = 2$. This follows that $|V_2| \geq 0$. First, consider $|V_2| = 0$. Then it implies that $|V_1| = 2$. Hence, we obtain $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = |V_1| = |V(G)| = 2$. Since G is a connected graph, it means that $G = K_2$. Next, consider that $|V_2| = 1$. Then it follows that $|V_1| = 0$ and $|V_0| > 0$. Let $V_2 = \{a\}$. Then V_2 is a convex set on G. Let $v \in V_0$. Since ρ is an OInCvRDF on G, there exist $x, y \in V(G) \setminus \{v\}$ such that $d_G(x, y) = d_G(x, v) + d_G(v, y)$, that is, V_0 is an interior vertex set on G. Since $V_0 \subseteq N_G(a)$, it is easy to see that $G = \langle\{a\}\rangle + \langle V_0\rangle$. Let $K_1 = \{a\}$ and $H = \langle V_0\rangle$. Thus, we have that $G = K_1 + H$ where $V(H)$ is an interior vertex set on G. Conversely, assume that $G \in \{K_2, K_1 + H\}$ where $V(H)$ is an interior vertex set on G. Suppose that $G = K_2$. Then it is clear that G does not have an interior vertex set. Thus, $|V_0| = 0$. By Theorem 3.5, $|V_2| = 0$. Hence, we get $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = |V_1| = 2$. Now, suppose that $G = K_1 + H$ where $V(H)$ is an interior vertex set on H. Let $V_0 = V(H)$, $V_1 = \varnothing$ and $V_2 = V(K_1)$. By construction, we have $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = 2|V(K_1)| = 2$, and so, (*ii*) is satisfied. This completes the proof. □

Corollary 3.15. *If $G = W_n$ (wheel graph) where $n \geq 4$, then $\widetilde{\gamma}_{InCvR}(G) = 2$.*

Proof. Let $G = W_n$ where $n \geq 4$. Then $G = K_1 + C_n$ where $n \geq 4$. Since $V(C_n)$ is an interior vertex set on G for $n \geq 4$, by Theorem 3.14(*ii*), it follows that $\widetilde{\gamma}_{InCvR}(G) = 2$. □

Theorem 3.16. *Let G be a connected non-complete graph of order n and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Then every cut-vertex $x \in V(G)$, $x \in V_1 \cup V_2$.*

Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Assume for a moment that there exists a cut vertex $x \in V(G)$ such that $x \notin V_1 \cup V_2$. Let G^1 and G^2 be the components of $G - \{x\}$. Let $x_1 \in (V_1 \cup V_2) \cap V(G^1)$ and $x_2 \in (V_1 \cup V_2) \cap V(G^2)$ such that for every $z \in (V_0 \cap (V(G^1) \cup V(G^2)))$, z is an interior vertex on G. Since x is a cut vertex on G, it follows that every x_1-x_2 shortest path must contain x. Thus, we have that $[x_1, x, x_2]$ is a x_1-x_2 geodesic on G. Hence, $(V_1 \cup V_2) \setminus \{x\}$ is not a convex set on G, a contradiction. Therefore, every cut-vertex $x \in V(G)$, $x \in V_1 \cup V_2$. □

Corollary 3.17. *If $G = P_n$ (path graph) where $n \geq 1$, then $\widetilde{\gamma}_{InCvR}(G) = n$.*

Proof. Let $G = P_n$ where $n \geq 1$ and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Let $n \leq 2$. Then by Theorem 3.14, $\widetilde{\gamma}_{InCvR}(G) = n$. Let $n \geq 3$ and let $G = P_n = [p_1, p_2, ..., p_n]$. Since $\{p_2, p_3, ..., p_{n-1}\}$ is a set of cut-vertex in G, it follows that $\{p_2, p_3, ..., p_{n-1}\} \subseteq V_1 \cup V_2$. However, since p_1 and p_n are not interior vertices of G, it also follows that $\{p_1, p_n\} \subseteq V_1 \cup V_2$. Since ρ is a $\widetilde{\gamma}_{InCvR}$-function on G, we have that $\{p_1, p_2, ..., p_n\} = V_1$ and $V_0 = V_2 = \varnothing$. Hence, $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. This proves the assertion. □

Proposition 3.18. *Let G be a connected non-complete graph of order n and let $\rho = (V_0, V_1, V_2)$ be an OInCvRDF on G. If for every $u, v \in V_1 \cup V_2$, $d_G(u, v) > 1$, then every x in u-v geodesic, $x \in V_1 \cup V_2$.*

Proof. Let $\rho = (V_0, V_1, V_2)$ be an OInCvRDF on G. Assume that for every $u, v \in V_1 \cup V_2$, $d_G(u, v) > 1$. Suppose, for the contrary that, there exists x in u-v geodesic such that $x \notin V_1 \cup V_2$. Then $x \in V_0$. This follows that $d_{\langle V_1 \cup V_2 \rangle}(u, v) = \infty$, a contradiction to the convexity of $V_1 \cup V_2$. Therefore, for every x in u-v geodesic, $x \in V_1 \cup V_2$. This completes the proof. □

Corollary 3.19. *If $G = C_n$ (cycle graph) where $n \geq 3$, then $\widetilde{\gamma}_{InCvR}(G) = n$.*

Proof. Let $G = C_n$ where $n \geq 3$ and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. If $n = 3$, then by Theorem 3.10, $\widetilde{\gamma}_{InCvR}(G) = 3$. Let $n \geq 4$ and let $G = C_n = [c_1, c_2, ..., c_n, c_1, ...]$. Then $V(G)$ is an interior vertex set on G. Let $\{c_2, c_{n-1}\} \subseteq V_2$. By Theorem 3.18, we have that $\{c_3, c_4, ..., c_{n-2}\} \subseteq V_1 \cup V_2$. This follows that $\{c_1, c_n\} = V_0$. Now, since ρ is a $\widetilde{\gamma}_{InCvR}$-function on G, it is forced that $\{c_3, c_4, ..., c_{n-2}\} = V_1$. This implies that $|V_0| = 2 = |V_2|$. By Theorem 3.8, we therefore have $\widetilde{\gamma}_{InCvR}(G) = n$. This proves the assertion. □

Theorem 3.20. *Let G be a connected graph of order n and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G for which $\widetilde{\gamma}_{InCvR}(G) < n$. Then $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\gamma}_{InCv}(G)+1$ if and only if $V_2 = \{v\}$ such that $V_0 \subseteq N_G(v)$ and V_0 is an interior vertex set on G.*

Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G for which $\widetilde{\gamma}_{InCvR}(G) < n$. Then $|V_0| \neq 0$ by Remark 3.1. By Theorem 3.5, $|V_2| \neq 0$. Assume that $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\gamma}_{InCv}(G) + 1$. Suppose, for the contrary that, $|V_2| \geq 2$. Let $v \in V_2$ such that $V_0 \subseteq N_G(v)$ and V_0 is an interior vertex set on G. Let $I \subseteq V(G)$ be a $\widetilde{\gamma}_{InCv}$-set on G. Then I is an outer-interior convex dominating set on G. By Theorem 3.2, we let $I = V_1 \cup V_2$. Hence, we have that $\widetilde{\gamma}_{InCv}(G) = |I| = |V_1| + |V_2| = |V_1| + 2|V_2| - |V_2| = \widetilde{\omega}_G^{InCvR}(\rho) - |V_2| = \widetilde{\gamma}_{InCvR}(G) - |V_2|$. Accordingly, $\widetilde{\gamma}_{InCv}(G) = \widetilde{\gamma}_{InCvR}(G) - |V_2|$ where $|V_2| \geq 2$. This is a contradiction. Hence, it suffices to say that $V_2 = \{v\}$ such that $V_0 \subseteq N_G(v)$ and V_0 is an interior vertex set on G. Conversely, assume that $V_2 = \{v\}$ such that $V_0 \subseteq N_G(v)$ and V_0 is an interior vertex set on G. Let $I \subset V(G)$ be a $\widetilde{\gamma}_{InCv}$-set on G. Then I is an outer-interior convex dominating set on G. By Theorem 3.2, we let $I = V_1 \cup V_2$. Then it follows that $\widetilde{\gamma}_{InCv}(G) \leq |I| = |V_1| + |V_2|$. Now, since ρ is a $\widetilde{\gamma}_{InCvR}$-function on G, it implies that $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1|+2|V_2| = |V_1|+|V_2|+|V_2| = |I|+|V_2| \geq \widetilde{\gamma}_{InCv}(G)+|V_2|$. Consequently, we get $\widetilde{\gamma}_{InCvR}(G) \geq \widetilde{\gamma}_{InCv}(G)+1$. It is worth noting that $\widetilde{\gamma}_{InCvR}(G) \leq \widetilde{\gamma}_{InCv}(G)+1$ whenever $|V_2| = 1$. Therefore, $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\gamma}_{InCv}(G)+1$. This completes the proof. □

Theorem 3.21. *Let $G \notin \{K_2, K_1 + H\}$ be a connected graph where $V(H)$ is an interior vertex set on G and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Then $V_1 = \varnothing$ and $\widetilde{\gamma}_{InCvR}(G) = 4$ if and only if there exists a convex dominating set $\{x, y\}$ such that $V(G) \setminus \{x, y\} = V_0$ is an interior vertex set on G.*

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Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Assume that $V_1 = \varnothing$ and $\widetilde{\gamma}_{InCvR}(G) = 4$. Then $\widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = 2|V_2| = 4$. This follows that $|V_2| = 2$. Let $V_2 = \{x, y\}$. By Theorem 3.2, $\{x, y\}$ is an outer-interior convex dominating set on G. This means that $\{x, y\}$ is a convex dominating set on G such that $V(G) \setminus \{x, y\} = V_0$ is an interior vertex set on G. Conversely, assume that there exists a convex dominating set $\{x, y\}$ such that $V(G)\setminus\{x, y\} = V_0$ is an interior vertex set on G. Then $\{x, y\}$ is an outer-interior convex dominating set on G. If $V_1 = \{x, y\}$ and $V_2 = \varnothing$, then $G = K_2$, a contradiction. If $V_1 = \{x\}$ and $V_2 = \{y\}$, then it is a contradiction since ρ is a $\widetilde{\gamma}_{InCvR}$-function on G. Thus, it follows that $V_1 = \varnothing$ and $V_2 = \{x, y\}$. Hence, $\widetilde{\gamma}_{InCvR}(G) \leq |V_1| + 2|V_2| = 2|V_2| = 2(2) = 4$. Since $G \notin \{K_2, K_1 + H\}$ is a connected graph where $V(H)$ is an interior vertex set on G, by Theorem 3.14(ii), it follows that $|V_2| \geq 2$. And so, we get $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = 2|V_2| \geq 4$. Therefore, we end up with $\widetilde{\gamma}_{InCvR}(G) = 4$. This proves the assertion. □

Theorem 3.22. *Let G be a connected graph of order n. Then $\widetilde{\gamma}_{InCv}(G) = \widetilde{\gamma}_{InCvR}(G)$ if and only if $\widetilde{\gamma}_{InCv}(G) = n$.*

Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Assume that $\widetilde{\gamma}_{InCv}(G) = \widetilde{\gamma}_{InCvR}(G)$. Then it follows that $|V_1| + |V_2| = |V_1| + 2|V_2| = \widetilde{\omega}_G^{InCvR}(\rho)$. This follows that $|V_2| = 0$. By Theorem 3.5, $|V_0| = 0$. Hence, we obtain $\widetilde{\gamma}_{InCv}(G) = |V_1| = |V(G)| = n$. Conversely, assume that $\widetilde{\gamma}_{InCv}(G) = n$. By Theorem 3.7, $n = \widetilde{\gamma}_{InCv}(G) \leq \widetilde{\gamma}_{InCvR}(G) \leq n$. Consequently, we have $\widetilde{\gamma}_{InCv}(G) = \widetilde{\gamma}_{InCvR}(G)$. This proves the assertion. □

Definition 3.23. Let G be a connected graph. Then G is called an *outer-interior convex Roman graph* if and only if $2\widetilde{\gamma}_{InCv}(G) = \widetilde{\gamma}_{InCvR}(G)$.

Using the concept of Definition 3.23, the following characterization is realized.

Theorem 3.24. *Let G be a connected graph and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Then the following are logically equivalent:*

(i) G is an outer-interior convex Roman graph;

(ii) G has a $\widetilde{\gamma}_{InCvR}$-function ρ such that $V_1 = \varnothing$; and

(iii) G has a $\widetilde{\gamma}_{InCvR}$-function ρ such that V_2 is a $\widetilde{\gamma}_{InCv}$-set on G.

Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Assume that G is an outer-interior convex Roman graph. Then $2\widetilde{\gamma}_{InCv}(G) = 2(|V_1| + |V_2|) = 2|V_1| + 2|V_2| = \widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2|$. Consequently, we get $|V_1| = 0$ and so, $V_1 = \varnothing$. Thus $(i) \implies (ii)$. Now, assume that G has a $\widetilde{\gamma}_{InCvR}$-function ρ such that $V_1 = \varnothing$. By Theorem 3.4, it implies that V_2 is a $\widetilde{\gamma}_{InCv}$-set on G. Hence, $(ii) \implies (iii)$. Further, assume that G has a $\widetilde{\gamma}_{InCvR}$-function ρ such that V_2 is a $\widetilde{\gamma}_{InCv}$-set on G. Then we have that $\widetilde{\gamma}_{InCv}(G) = |V_2|$. By Theorem 3.4, $V_1 = \varnothing$. This follows that $\widetilde{\gamma}_{InCvR}(G) = \widetilde{\omega}_G^{InCv}(\rho) = |V_1| + 2|V_2| = 2|V_2| = 2\widetilde{\gamma}_{InCv}(G)$. Therefore, $(iii) \implies (i)$. This completes the proof. □

The next corollary is immediate from Theorem 3.24.

Corollary 3.25. *Let G be a connected graph. If $\widetilde{\gamma}_{In}(G) = 1$, then G is an outer-interior convex Roman graph with $\widetilde{\gamma}_{InCvR}(G) = 2$.*

Proof. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{InCvR}$-function on G. Assume $\widetilde{\gamma}_{In}(G) = 1$. Then it is easy to see that $V_1 = \varnothing$ and $\widetilde{\gamma}_{In}(G) = |V_2| = 1$. Since ρ is a $\widetilde{\gamma}_{InCvR}$-function on G, by Theorem 3.4, it follows that V_2 is a $\widetilde{\gamma}_{InCv}$-set on G, that is, $\widetilde{\gamma}_{InCv}(G) = |V_2|$. Accordingly, by Theorem 3.24, G is an outer-interior convex Roman graph with $\widetilde{\gamma}_{InCvR}(G) = 2$. This completes the proof. □

Theorem 3.26. *Let G and H be connected noncomplete graphs. Then*

$$\widetilde{\gamma}_{InCvR}(G+H) = \begin{cases} 2, & \text{if } \widetilde{\gamma}_{In}(G) = 1 \text{ or } \widetilde{\gamma}_{In}(H) = 1, \\ 4, & \text{if } \widetilde{\gamma}_{In}(G) \neq 1 \text{ and } \widetilde{\gamma}_{In}(H) \neq 1. \end{cases}$$

Proof. It is clear that $G+H \neq K_1$. Then by Theorem 3.14(i), it follows that $\widetilde{\gamma}_{InCvR}(G+H) \geq 2$. Suppose that $\widetilde{\gamma}_{In}(G) = 1$ or $\widetilde{\gamma}_{In}(H) = 1$. Then it follows that $\widetilde{\gamma}_{InCvR}(G+H) \leq 2$. Consequently, $\widetilde{\gamma}_{InCvR}(G+H) = 2$. Now, suppose that $\widetilde{\gamma}_{In}(G) \neq 1$ and $\widetilde{\gamma}_{In}(H) \neq 1$. Then it implies that $\widetilde{\gamma}_{InCvR}(G+H) \geq 4$. Let $x \in V(G)$ and $y \in V(H)$. Then $xy \in E(G+H)$ and so, $\{x, y\}$ is a convex set on $G+H$. Moreover, $V(G) \subseteq N_G(x)$ and $V(H) \subseteq N_H(y)$. This follows that $V(G+H) = N_{G+H}[\{x, y\}]$. Since G and H are connected noncomplete graphs, it means that $V(G+H)$ is an interior vertex set on $G+H$, and this indicates that $V(G+H) \setminus \{x, y\}$ is also an interior vertex set on $G+H$. Hence, we let $V_0 = V(G+H) \setminus \{x, y\}$, $V_1 = \varnothing$ and $V_2 = \{x, y\}$. Then $\rho = (V_0, V_1, V_2)$ is an OInCvRDF on $G+H$. Hence, we obtain $\widetilde{\gamma}_{InCvR}(G+H) \leq \widetilde{\omega}_G^{InCvR}(\rho) = |V_1| + 2|V_2| = 2|V_2| = 2|\{x, y\}| = 4$. Accordingly, $\widetilde{\gamma}_{InCvR}(G+H) = 4$. This proves the assertion. □

4 Conclusion

This paper introduced a new parameter of convex Roman domination in graphs called the outer-interior convex Roman dominating function. The study investigated the combinatorial properties of outer-interior convex Roman dominating functions and constructed some characterizations. Moreover, this research obtained some bounds for any connected graph classes and exact values for some special graphs. This study recommends characterizing the outer-interior convex Roman dominating function under some binary operations of two graphs, such as corona, Cartesian, and lexicographic products.

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