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Equi-integrability in the Monotone and the Dominated Convergence Theorems for the McShane Integral

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Abstract: In this paper, we prove the equi-integrability of the sequences of functions in the monotone and dominated convergence theorems for the McShane integral.

Keywords/Phrases: McShane integral, equi-integrable, uniformly gauge Cauchy, Monotone Convergence Theorem, Dominated Convergence Theorem

1 Introduction

The Lebesgue integral is noted for its powerful convergence theorems - the Monotone and Dominated Convergence theorems. Lee in [3] prove these two convergence theorems for the Henstock integral. By following similar proofs, one can also prove their corresponding versions for the McShane integral.

In this paper, we show that in each of these theorems, the integrability of the sequence considered is in fact uniform in the sense that, given $\epsilon > 0$, then *the same gauge function* is valid simultaneously for the integrability of all the functions in the sequence. This concept, called *equi-integrability*, is due to Jaroslav Kurzweil [2].

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2 Preliminary Concepts and Known Results

We begin by defining important concepts, such as the *McShane integral* and *McShane equi-integrability*, and stating some of the known results we need for the main results. Readers who seek to know the details of this integral are referred to [1] and [8].

Definition 2.1 A function $f : [a, b] \to \mathbb{R}$ is said to be *McShane integrable to a real number A on* $[a, b]$ if for any $\epsilon > 0$, there exists a gauge $\delta(\xi) > 0$ on $[a, b]$ such that for any McShane δ-fine division $D = \{([u, v], \xi)\}$ of $[a, b]$, we have

$$\left|(D)\sum f(\xi)(v-u) - A\right| < \epsilon.$$

If $f : [a, b] \to \mathbb{R}$ is McShane integrable to A on $[a, b]$, then we write

$$A = (\mathcal{M})\int_a^b f.$$

By a McShane δ-fine division $D = \{([u, v]; \xi)\}$ of $[a, b]$ we mean that $[u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$.

Definition 2.2 A sequence $\{f_n\}_{n=1}^{\infty}$ of McShane integrable functions on $[a, b]$ is *McShane equi-integrable* (or simply *equi-integrable*) on $[a, b]$ if for any $\epsilon > 0$, there exists a gauge $\delta(\xi) > 0$ on $[a, b]$ such that for any McShane δ-fine division $D = \{([u, v], \xi)\}$ of $[a, b]$, we have

$$\left|(D)\sum f_n(\xi)(v-u) - (\mathcal{M})\int_a^b f_n\right| < \epsilon, \quad \text{for all } n.$$

Definition 2.2 requires the existence of a gauge thats works uniformly for the integrability of all the functions f_n.

It should be noted that if for every $\epsilon > 0$ there exist $\delta(\xi) > 0$ and $N \in \mathbb{N}$ such that

$$\left|(D)\sum f_n(\xi)(v-u) - (\mathcal{M})\int_a^b f_n\right| < \epsilon$$

for all $n \geq N$ and McShane δ-fine divisions $D = \{([u,v],\xi)\}$ of $[a,b]$, then $\{f_n\}_{n=1}^{\infty}$ is McShane equi-integrable.

Lemma 2.3 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of McShane integrable functions on $[a,b]$. If $\{f_n\}_{n=1}^{\infty}$ is equi-integrable on $[a,b]$ and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for each $x \in [a,b]$, then the sequence $\left\{(\mathcal{M})\int_a^b f_n\right\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$.

Proof: Let $\epsilon > 0$. By equi-integrability of $\{f_n\}_{n=1}^{\infty}$, there exists $\delta(\xi) > 0$ such that for each n

$$\left|(D)\sum f_n(\xi)(v-u) - (\mathcal{M})\int_a^b f_n\right| < \epsilon \tag{9}$$

whenever $D = \{([u,v],\xi)\}$ is a McShane δ-fine division of $[a,b]$.

Fix a McShane δ-fine division $D' = \{([u,v],\xi)\}$ of $[a,b]$. Since

$$\lim_{n\to\infty} f_n(x) = f(x),$$

for each tag ξ in D' there exists a positive integer $N(\xi)$ such that for each $n \geq N(\xi)$, we have

$$\left|f_n(\xi) - f(\xi)\right| < \epsilon.$$

Let $N = \max\{N(\xi) : \xi$ is a tag point in $D'\}$. Then for each $n \geq N$

$$\begin{aligned}
&\left|(\mathcal{M})\int_a^b f_n - (D')\sum f(\xi)(v-u)\right| \\
&\leq \left|(\mathcal{M})\int_a^b f_n - (D')\sum f_n(\xi)(v-u)\right| \\
&\qquad + \left|(D')\sum f_n(\xi)(v-u) - (D')\sum f(\xi)(v-u)\right| \\
&< \epsilon + \epsilon \cdot (b-a) \\
&= \epsilon[1+(b-a)].
\end{aligned}$$

Thus, for each $n, m \geq N$

$$\begin{aligned}
&\left|(\mathcal{M})\int_a^b f_n - (\mathcal{M})\int_a^b f_m\right| \\
&\qquad \leq \left|(\mathcal{M})\int_a^b f_n - (D')\sum f(\xi)(v-u)\right| \\
&\qquad + \left|(D')\sum f(\xi)(v-u) - (\mathcal{M})\int_a^b f_m\right| \\
&\qquad < \epsilon[2+2(b-a)].
\end{aligned}$$

This shows that $\left\{(\mathcal{M})\int_a^b f_n\right\}$ is Cauchy in $\mathbb{R}$. □

The following is a simple convergence theorem involving an equi-integrable sequence of functions.

Theorem 2.4 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of McShane integrable functions on $[a,b]$. If $\{f_n\}_{n=1}^{\infty}$ is equi-integrable on $[a,b]$ and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for each $x \in [a,b]$, then f is McShane integrable and

$$\lim_{n\to\infty} (\mathcal{M}) \int_a^b f_n = (\mathcal{M}) \int_a^b f.$$

Proof: Let $\epsilon > 0$. By Lemma 2.3, the sequence $\left\{(\mathcal{M}) \int_a^b f_n\right\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. Hence, $\left\{(\mathcal{M}) \int_a^b f_n\right\}$ is convergent in $\mathbb{R}$. Let

$$A = \lim_{n\to\infty} (\mathcal{M}) \int_a^b f_n. \tag{10}$$

We claim that $A = (\mathcal{M}) \int_a^b f$. There exists $\delta(\xi) > 0$ such that for all n, inequality (9) holds whenever $D = \{([u,v],\xi)\}$ is a McShane δ-fine division of $[a,b]$. Applying (10),we obtain

$$\left|(D) \sum f(\xi)(v-u) - A\right| < \epsilon,$$

for all McShane δ-fine divisions $D = \{([u,v],\xi)\}$ of $[a,b]$. This shows that f is McShane integrable on $[a,b]$ and

$$(\mathcal{M}) \int_a^b f = A = \lim_{n\to\infty} (\mathcal{M}) \int_a^b f_n. \qquad \square$$

The following is a version of the Monotone Convergence Theorem. This theorem, and its proof, for the Henstock integral is well-known (see Lee [3]).

Theorem 2.5 *Let $\{f_n\}_{n=1}^{\infty}$ be an increasing sequence of McShane integrable functions on $[a,b]$ and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for each $x \in [a,b]$. *If* $\lim_{n\to\infty} (\mathcal{M}) \int_a^b f_n = A$, *then* f *is McShane integrable on* $[a,b]$ *and*

$$(\mathcal{M}) \int_a^b f = A = \lim_{n\to\infty} (\mathcal{M}) \int_a^b f_n.$$

Definition 2.6 [8] A sequence $\{f_n\}_{n=1}^{\infty}$ of McShane integrable functions on $[a,b]$ is said to be *uniformly gauge Cauchy* on $[a,b]$ if for any $\epsilon > 0$, there exists a gauge $\delta(\xi) > 0$ and a positive integer N such that for each $n, m \geq N$, we have

$$\left|(D) \sum f_n(\xi)(v-u) - (D) \sum f_m(\xi)(v-u)\right| < \epsilon$$

whenever $D = \{([u,v],\xi)\}$ is a McShane δ-fine division of $[a,b]$.

Theorem 2.7 [8] *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of McShane integrable functions on* $[a,b]$. *The following are equivalent:*

(*i*) $\{f_n\}_{n=1}^{\infty}$ *is uniformly gauge Cauchy.*

(*ii*) $\left\{(\mathcal{M}) \int_a^b f_n\right\}_{n=1}^{\infty}$ *converges and* $\{f_n\}_{n=1}^{\infty}$ *is equi-integrable on* $[a,b]$.

3 Results

First, we state and prove the following Lemmas.

Lemma 3.1 *If* $\{\varphi_n\}_{n=1}^{\infty}$ *is a decreasing sequence of McShane integrable functions on* $[a,b]$ *and for any* $t \in [a,b]$,

$$\lim_{n\to\infty} \varphi_n(t) = 0,$$

then

$$\lim_{n\to\infty} (\mathcal{M}) \int_a^b \varphi_n = 0.$$

Proof: The sequence $\{-\varphi_n\}_{n=1}^{\infty}$ is increasing and

$$\lim_{n\to\infty}(-\varphi_n(t)) = 0.$$

By Theorem 2.5,

$$\begin{aligned} -\left(\lim_{n\to\infty}(\mathcal{M})\int_a^b \varphi_n\right) &= \lim_{n\to\infty}(\mathcal{M})\int_a^b(-\varphi_n) \\ &= (\mathcal{M})\int_a^b 0 \\ &= 0. \end{aligned}$$

Hence,

$$\lim_{n\to\infty}(\mathcal{M})\int_a^b \varphi_n = 0. \qquad \square$$

Lemma 3.2 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of McShane integrable functions on $[a,b]$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ for each $t \in [a,b]$. Suppose $\{\varphi_n\}_{n=1}^{\infty}$ is a decreasing sequence of McShane integrable functions on $[a,b]$ such that $\lim_{n\to\infty}\varphi_n(t) = 0$; and for each n, there exists a positive integer M_n such that for each $x \in [a,b]$, we have*

$$\left|f_i(x) - f_j(x)\right| \le \varphi_n(x)$$

for each $i, j \ge M_n$. Then $\{f_n\}_{n=1}^{\infty}$ is equi-integrable on $[a,b]$ and f is McShane integrable on $[a,b]$ with

$$(\mathcal{M})\int_a^b f = \lim_{n\to\infty}(\mathcal{M})\int_a^b f_n.$$

Proof: By Lemma 3.1,

$$\lim_{n\to\infty}(\mathcal{M})\int_a^b \varphi_n = 0.$$

We claim that $\{f_n\}_{n=1}^{\infty}$ is uniformly gauge Cauchy. Let $\epsilon > 0$ be given. Then there exists a positive integer N such that

$$\left|(\mathcal{M})\int_a^b \varphi_N\right| < \frac{\epsilon}{2}.$$

By hypothesis, corresponding to N, there exists a positive integer M_N such that for each $x \in [a,b]$ and $i,j \geq M_N$, we have

$$|f_i(x) - f_j(x)| \leq \varphi_N(x).$$

Since each φ_n is McShane integrable on $[a,b]$, there exists $\delta_n(\xi) > 0$ such that whenever $D = \{([u,v],\xi)\}$ is a McShane δ_n-fine division of $[a,b]$, we have

$$\left|(D)\sum \varphi_n(\xi)(v-u) - (\mathcal{M})\int_a^b \varphi_n\right| < \frac{\epsilon}{2}.$$

We may assume that $\delta_n \geq \delta_{n+1}$ for each n. Note that $\varphi_n \geq 0$ for each n. Define $\delta(\xi) = \delta_N(\xi)$ for each $\xi \in [a,b]$, and let $D = \{([u_k,v_k],\xi_k) : k = 1,2,\ldots,r\}$ be any McShane δ-fine division of $[a,b]$. Then

$$\begin{aligned}
&\sum_{k=1}^{r} \varphi_N(\xi_k)(v_k - u_k) \\
&\leq \left|\sum_{k=1}^{r} \varphi_N(\xi_k)(v_k - u_k) - (\mathcal{M})\int_a^b \varphi_N\right| + \left|(\mathcal{M})\int_a^b \varphi_N\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, for each $i, j \geq M_N$

$$\begin{aligned}\left|\sum_{k=1}^{r} f_i(\xi_k)(v_k - u_k) - \sum_{k=1}^{r} f_j(\xi_k)(v_k - u_k)\right| \\ &\leq \sum_{k=1}^{r} \left\{\left|f_i(\xi_k) - f_j(\xi_k)\right|(v_k - u_k)\right\} \\ &\leq \sum_{k=1}^{r} \varphi_N(\xi_k)(v_k - u_k) \\ &= \epsilon.\end{aligned}$$

Consequently, $\{f_n\}_{n=1}^{\infty}$ is uniformly gauge Cauchy. By Theorem 2.7, $\left\{(\mathcal{M})\int_a^b f_n\right\}_{n=1}^{\infty}$ converges and $\{f_n\}_{n=1}^{\infty}$ is equi-integrable. The assertion

$$\lim_{n\to\infty}(\mathcal{M})\int_a^b f_n = (\mathcal{M})\int_a^b f$$

follows from Theorem 2.4. □

Now follows the first desired result of this study.

Theorem 3.3 (Monotone Convergence Theorem) *Let $\{f_n\}_{n=1}^{\infty}$ be an increasing sequence of McShane integrable functions on $[a,b]$ such that*

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ , for each } x \in [a,b].$$

If $\sup\left\{(\mathcal{M})\int_a^b f_n : n \in \mathbb{N}\right\} < \infty$, then $\{f_n\}_{n=1}^{\infty}$ is equi-integrable on $[a,b]$ and f is McShane integrable on $[a,b]$ and

$$\lim_{n\to\infty}(\mathcal{M})\int_a^b f_n = (\mathcal{M})\int_a^b f.$$

Proof: For each $t \in [a,b]$, the sequence $\{f_n(t)\}_{n=1}^{\infty}$ converges and thus, is bounded. Hence, for each $t \in [a,b]$, there exists $K(t)$ such that $|f_n(t)| \leq K(t)$ for each n. Then for $i \geq j$ and $t \in [a,b]$,

$$f_i(t) - f_j(t) = |f_i(t) - f_j(t)| \leq |f_i(t)| + |f_j(t)| \leq 2K(t).$$

For $n \leq m$, let

$$\varphi_{n,m} = \max\{f_i - f_j : n \leq j \leq i \leq m\}.$$

Since f_n are McShane integrable, $\varphi_{n,m}$ is also McShane integrable on $[a,b]$. For a fix n,

$$\begin{aligned} \varphi_{n,m} &= \max\left\{f_i - f_j : n \leq j \leq i \leq m\right\} \\ &\leq \max\left\{f_i - f_j : n \leq j \leq i \leq m+1\right\} \\ &= \varphi_{n,m+1}. \end{aligned}$$

Hence, for each n, the sequence $\{\varphi_{n,m} \ldots\}_{m=1}^{\infty}$ is increasing and

$$\begin{aligned} \lim_{m\to\infty} \varphi_{n,m} &= \lim_{m\to\infty} \max\left\{f_i - f_j : n \leq j \leq i \leq m\right\} \\ &= \sup\left\{f_i - f_j : n \leq j \leq i\right\}. \end{aligned}$$

Let $\varphi_n = \sup\left\{f_i - f_j : n \leq j \leq i\right\}$. For each n,

$$\begin{aligned} \varphi_n &= \sup\left\{f_i - f_j : n \leq j \leq i\right\} \\ &\geq \sup\left\{f_i - f_j : n+1 \leq j \leq i\right\} \\ &= \varphi_{n+1} \end{aligned}$$

Thus, $\{\varphi_n\}_{n=1}^{\infty}$ is a decreasing sequence of non-negative McShane integrable functions on $[a,b]$ and

$$\lim_{n\to\infty} \varphi_n(t) = \lim_{n\to\infty} \sup\left\{f_i(t) - f_j(t) : n \leq j \leq i\right\} = 0.$$

For each n and $x \in [a,b]$,

$$\begin{aligned} \big|f_i(x) - f_j(x)\big| &\leq \sup\big\{\big|f_i(t) - f_j(t)\big| : n \leq j \leq i\big\} \\ &= \sup\big\{f_i(t) - f_j(t) : n \leq j \leq i\big\} \\ &= \varphi_n(x), \end{aligned}$$

for each $i, j \geq n$. The conclusions follow immediately from Theorem 3.2. $\square$

If $\{f_n\}_{n=1}^{\infty}$ is decreasing, then by considering the sequence $\{-f_n\}_{n=1}^{\infty}$, an analogous result also holds for decreasing sequence.

We now state and prove the Dominated Convergence Theorem.

Theorem 3.4 (Dominated Convergence Theorem) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of McShane integrable functions on $[a,b]$ such that*

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ , for each } x \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is McShane integrable on $[a,b]$ and $|f_n - f_m| \leq g$ for each n, m, then $\{f_n\}_{n=1}^{\infty}$ is equi-integrable on $[a,b]$ and f is McShane integrable on $[a,b]$ with

$$(\mathcal{M})\int_a^b f = \lim_{n\to\infty} (\mathcal{M})\int_a^b f_n.$$

Proof: For any $n \leq m$, let

$$\varphi_{n,m} = \max\big\{|f_i - f_j| : n \leq i \leq j \leq m\big\}.$$

Then each $\varphi_{n,m}$ is McShane integrable on $[a,b]$. For a fix n,

$$\begin{aligned} \varphi_{n,m} &= \max\big\{|f_i - f_j| : n \leq i \leq j \leq m\big\} \\ &\leq \max\big\{|f_i - f_j| : n \leq i \leq j \leq m+1\big\} \\ &= \varphi_{n,m+1}. \end{aligned}$$

Hence, for each n, the sequence $\{\varphi_{n,m}\}_{m=1}^{\infty}$ is increasing and converges to $\varphi_n = \sup\left\{|f_i - f_j| : n \le i \le j\right\}$. Note that $\varphi_n \le g$ and since g is McShane integrable, we have

$$\sup_m \left\{(\mathcal{M}) \int_a^b \varphi_{n,m}\right\} \le (\mathcal{M}) \int_a^b g < \infty.$$

By Monotone Convergence Theorem (Theorem 3.3), φ_n is McShane integrable on $[a, b]$ and

$$(\mathcal{M}) \int_a^b \varphi_n = \lim_{m\to\infty} (\mathcal{M}) \int_a^b \varphi_{n,m} \le (\mathcal{M}) \int_a^b g$$

for each $n \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$

$$\begin{aligned} \varphi_n &= \sup\left\{|f_i - f_j| : n \le i \le j\right\} \\ &\ge \sup\left\{|f_i - f_j| : n+1 \le i \le j\right\} \\ &= \varphi_{n+1} \end{aligned}$$

so that $\{\varphi_n\}_{n=1}^{\infty}$ is a decreasing sequence of McShane integrable non-negative functions on $[a, b]$. Moreover, since $f_n(t) \to f(t)$ for $t \in [a, b]$, we have

$$\lim_{n\to\infty} \varphi_n(t) = \lim_{n\to\infty} \sup\left\{|f_i(t) - f_j(t)| : n \le i \le j\right\} = 0$$

for any $t \in [a, b]$. By Theorem 3.1,

$$\lim_{n\to\infty} (\mathcal{M}) \int_a^b \varphi_n = 0.$$

For each n and $x \in [a, b]$,

$$\left|f_i(x) - f_j(x)\right| \le \sup\left\{f_i(t) - f_j(t) : n \le j \le i\right\} = \varphi_n(x),$$

for each $i, j \ge n$. Theorem 3.2 finally yields the desired conclusions. □

The condition $|f_n - f_m| \le g$ for each n, m is equivalent to the usual dominated condition $|f_n| \le h$ for each n.

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